Exercise 7.1. ♣
Which of the following series is the Fourier series of a function $f \in L^2((-T,T))$ for an appropriate choice of $T > 0$?
(a) $\sum_{n=1}^{\infty} \frac{\cos(nx)}{\sqrt{n}}$.

(b) $\sum_{n=2}^{\infty} \frac{\sin(n\pi x)}{\sqrt{n}\log(n)}$.

(c) $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} + \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$.

(d) $\sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n\sqrt{n}}$.

Exercise 7.2.
Let $f, g \in C(\mathbb{R})$ be 2π-periodic continuous functions. Compute the Fourier coefficients of each of the following 2π-periodic functions in terms of the Fourier coefficients of f, g.
(a) $f_\tau(x) := f(x-\tau)$ for some $\tau \in \mathbb{R}$.

(b) $f \cdot g(x) := f(x)g(x)$.

(c) $f * g(x) := \int_{-\pi}^{\pi} f(x-t)g(t)\,dt$.

Exercise 7.3.
The goal of this exercise is to show that every function in $L^2((0,\pi);\mathbb{R})$ can be expressed as a real Fourier series of sines.
(a) Show that if $f \in L^2((-\pi,\pi);\mathbb{R})$ is odd then its Fourier coefficients $c_n(f)$ are purely imaginary and $c_0(f) = 0$;

(b) Show that if $f \in L^2((-\pi,\pi);\mathbb{R})$ is odd then its N-th Fourier partial sum satisfies

$$S_N f(x) = \sum_{n=1}^{N} 2i c_n(f) \sin(nx).$$

(c) Given $g \in L^2((0,\pi);\mathbb{R})$ show that $\tilde{S}_N g \to g$ in $L^2((0,\pi);\mathbb{R})$ as $N \to \infty$, where

$$\tilde{S}_N g(x) := \sum_{n=1}^{N} a_n(g) \sin(nx), \qquad a_n(g) := \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx)\,dx.$$

(d) Conclude that $\left\{ \sqrt{\frac{2}{\pi}} \sin(nx) \right\}_{n \in \mathbb{N}}$ is a Hilbert basis for $L^2((0,\pi);\mathbb{R})$.

Exercise 7.4.
Note that the Fourier coefficients $c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$ are in fact well-defined for any $f \in L^1((-\pi,\pi))$. However, the Fourier series of an L^1 function does not necessarily converge. The goal of this exercise is to show that if $c_n(f) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.
(a) For $f \in L^1((-\pi,\pi))$, we define

$$f_r(x) = \sum_{n\in\mathbb{Z}} c_n(f) r^{|n|} e^{inx}, \quad \text{for } 0 \le r < 1.$$

Show that f_r is well-defined for all $r \in [0,1)$ and can be written as a convolution:

$$f_r(x) = P_r * f(x) := \int_{-\pi}^{\pi} P_r(x-y) f(y)\,dy, \quad \text{where} \quad P_r(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|}e^{inx}.$$

Remark: P_r is known as the Poisson kernel.

(b) Show that

$$P_r(x) = \frac{1}{2\pi} \cdot \frac{1-r^2}{1-2r\cos(x)+r^2}.$$

(c) Show that the family $(P_r)_{0\le r<1}$ is an approximate identity. That is,

- $P_r \ge 0$ and $\int_{-\pi}^{\pi} P_r(x)\,dx = 1$ for all $r \in [0,1)$.
- for all $\delta > 0$ we have $\int_{\{\delta<|x|<\pi\}} P_r(x)\,dx \to 0$ as $r \to 1^-$.

(d) Show that f_r converges to f in $L^1((-\pi,\pi))$ as $r \to 1^-$.
Hint: Use exercise 13.7 from Analysis III.

(e) Conclude that if f satisfies $c_n(f) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.

Exercise 7.5.
Let $f \in L^1((-\pi,\pi))$ and let $c_n(f)$ be its Fourier coefficients.
(a) Show that if $\sum_{n\in\mathbb{Z}} |c_n(f)|^2 < \infty$, then in fact $f \in L^2((-\pi,\pi))$.
Hint: Show that the sequence of Fourier partial sums S_N is Cauchy in L^2 and use the previous exercise.

(b) Show that if $\sum_{n\in\mathbb{Z}} |c_n(f)| < \infty$, then in fact[a] $f \in C_{per}([-\pi,\pi])$.
Hint: Show that the sequence of Fourier partial sums S_N is Cauchy in the uniform norm and use the previous exercise.

[a]This is a slight abuse of notation. More precisely: there exists a (necessarily unique) continuous and periodic $\tilde{f}$ such that $\tilde{f} = f$ a.e.