Exercise 8.1. ♣
For each of the following functions defined on $[-\pi, \pi]$,

- $f_1(x) = \tan(\sin(x))$
- $f_2(x) = |x|^{2/3}$
- $f_3(x) = x$
- $f_4(x) = e^{-x^2}$
- $f_5(x) = |x|^{-1/2}$

answer the following questions using the theorems seen in class. If none of the convergence theorems applies, that's still a valid answer.
(a) Are the Fourier coefficients well-defined?

(b) Is it true that $S_N(f) \to f$ in L^2?

(c) Is it true that $S_N(f)(x) \to f(x)$ for all $x \in [-\pi, \pi]$? **Hint**: Recall Theorem 2.28.

(d) Is it true that $S_N(f) \to f$ in C_{per}? **Hint**: Recall Corollary 2.20.

Exercise 8.2.
(a) Construct $f\colon [-\pi, \pi] \to \mathbb{R}$ which is continuous, but not Hölder at $x = 0$.
Hint: try using $1/\log(x/2\pi)$.

(b) Let V be the vector space of sequences $f\colon \mathbb{N} \to \mathbb{R}$ such that

$$\|f\|_V := \Big(\sum_{k\geq 1} k^2 |f(k)|^2\Big)^{1/2} < \infty.$$

Can you choose a scalar product on V that makes V a Hilbert space?
Hint: try to construct an L^2 space over $\mathbb{N}$ with the right measure.

(c) Let V be the vector space of sequences $f\colon \mathbb{N} \setminus \{0\} \to \mathbb{R}$ such that

$$\|f\|_V := \sum_{k\geq 1} k|f(k)| < \infty.$$

Can you choose a scalar product on V that makes V a Hilbert space?

(d) Explain the difference between the following spaces of (real) functions and provide elements that fit in one but none of the others:

$$C_{per}([-\pi, \pi]; \mathbb{R}), \quad C^2_{per}([-\pi, \pi]; \mathbb{R}), \quad C((-\pi, \pi); \mathbb{R}), \quad C([-\pi, \pi]; \mathbb{R}).$$

Exercise 8.3.
Let $u : [a, b] \to \mathbb{C}$ be continuous and piecewise C^1 on a compact interval $[a, b] \subset \mathbb{R}$ with $u(a) = u(b) = 0$.
(a) Show that

$$\int_a^b |u(x)|^2 \, dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b |u'(x)|^2 \, dx \tag{1}$$

Remark: (1) is known as the Wirtinger inequality.

(b) For which functions does equality hold in (1)?

Exercise 8.4.
Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and integrable on $[-\pi, \pi]$.
(a) Show that

$$c_n(f) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \tfrac{\pi}{n}\big) e^{-inx} \, dx,$$

and hence

$$c_n(f) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \Big(f(x) - f\big(x + \tfrac{\pi}{n}\big) \Big) e^{-inx} \, dx.$$

(b) Now assume that f is Hölder continuous of order α, that is

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for some $0 < \alpha \leq 1$, some $C > 0$ and all x, h.
Show that $c_n(f)$ is of order $|n|^{-\alpha}$, i.e. for some $\tilde{C} > 0$ and all $n \in \mathbb{Z}$:

$$|c_n(f)| \leq \frac{\tilde{C}}{|n|^\alpha}.$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x},$$

where $0 < \alpha < 1$, is Hölder continuous of order α and satisfies $c_n(f) = n^{-\alpha}$ whenever $n = 2^k$.
Hint: Break the sum up as follows $f(x+h) - f(x) = \sum_{2^k \leq |h|^{-1}} \cdots + \sum_{2^k > |h|^{-1}} \cdots$ and use the fact that $|1 - e^{i\theta}| \leq |\theta|$ for any $\theta \in \mathbb{R}$.

Exercise 8.5.
Recall that the Dirichlet kernel satisfies $D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$, for all $n \geq 1$ and $x \in \mathbb{R}$.
(a) Show that

$$\int_0^{\pi} |D_n(x)|\, dx > 2 \sum_{j=0}^{n-1} \int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y}.$$

Hint: Use $|\sin(t)| \leq |t|$, then change variables and divide up the domain of integration.

(b) Show that for each $j \geq 0$ we have

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y} \geq \frac{c}{j+1},$$

for some (explicit) constant $c > 0$.

(c) Conclude that $\|D_n\|_{L^1(0,\pi)} \geq C \log n$ as $n \to \infty$ for some $C > 0$.
Hint: Recall the asymptotic behavior of the harmonic series: $H_n \coloneqq \sum_{k=1}^{n} \frac{1}{k} \geq \log n$.
Remark: This shows that D_n does not converge in L^1 as $n \to \infty$.