Exercise 9.1. ♣
Which of the following statements are true? Recall that $\hat{f}$ denotes the Fourier transform of a function f.
(a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of compactly supported continuous functions on $\mathbb{R}^d$. If $f_n \to f$ uniformly, then f is continuous and compactly supported.

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of compactly supported continuous functions on $\mathbb{R}^d$. If $f_n \to f$ uniformly, then f is continuous and satisfies $\lim_{|x|\to\infty} f(x) = 0$.

(c) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$.

(d) If f is compactly supported on $\mathbb{R}^d$, then $\hat{f} \in L^1(\mathbb{R}^d)$.

(e) If f is compactly supported and bounded on $\mathbb{R}^d$, then $\hat{f}$ is continuous and satisfies $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

(f) For $f \in L^1(\mathbb{R}^d)$ define $f_t(x) := f(x)\mathbf{1}_{\{|f(x)|\geq t\}}$ for $t > 0$. Then

$$\sup_{\xi\in\mathbb{R}^d} |\hat{f}_t(\xi)| \to 0 \text{ as } t \to \infty.$$

Exercise 9.2.
For the following PDEs of evolution type, try to find the most general solution of the form $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{-ikx}$ without worrying about convergence issues (i.e. we are looking for 2π-periodic solutions to the PDEs). For each PDE also write down a specific example solution which is not a constant.
Remark: The functions $\{u_k(t)\}_{k\in\mathbb{Z}}$ will of course depend on the initial conditions, in particular the Fourier coefficients of $u(0,\cdot)$ (and sometimes also of $\partial_t u(0,\cdot)$).
(a) $\partial_t u = \cos(t)\partial_{xx} u$

(b) $\partial_{tt} u - \partial_{xx} u = 0$

(c) $\partial_t u = \frac{1}{1+t^2} u + \partial_{xx} u$

(d) $\partial_t u = \partial_{xx} u + 1$

Exercise 9.3.
Consider the following evolution problem with periodic boundary conditions:

$$\begin{cases} i\partial_t u + \partial_{xx} u = 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } 2\pi\text{-periodic } f \in C^\infty(\mathbb{R}). \end{cases}$$

Remark: This PDE is a version of the Schrödinger equation.
(a) Explain why solutions cannot be purely real-valued, unless they are constant.

(b) Explain why, for each fixed large $N \in \mathbb{N}$, we have $\sup_{k\in\mathbb{Z}} |k|^N |c_k(f)| < \infty$.

(c) Find the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on the Fourier coefficients of f.

(d) Show that the formal solution is in fact a true solution and is C^∞ in both variables.
Hint: You need to show that the Fourier coefficients $\{c_k(\partial_t^m \partial_x^n u(t,\cdot))\}$ are summable. This follows from the decay of the $\{c_k(f)\}$.

(e) Show that we found the only possible solution: if v is a solution of the problem which is C^2_{per} in space and C^1 in time, then $u = v$.
Hint: Argue exactly as in the proof of uniqueness for the heat equation.

(f) Find the explicit solution u in the case $f(x) = 2\cos(3x)$.

(g) Does this equation enjoy the "smoothing effect" of the heat equation?
Hint: Observe that the size of u_k and the size of $c_k(f)$ are comparable: do we expect regularization?

Exercise 9.4.
Given $f \in L^2((-\pi,\pi))$ as initial data, consider the associated periodic solution to the heat equation defined by

$$u(t,x) := \sum_{k\in\mathbb{Z}} c_k(f) e^{ikx-k^2t}, \quad \text{for all } x \in \mathbb{R},\ t > 0.$$

(a) Show that $u \in C^\infty((0,\infty)\times\mathbb{R})$ and u solves the heat equation

$$\partial_t u(t,x) = \partial_{xx} u(t,x), \qquad \text{for all } x \in \mathbb{R},\ t > 0. \tag{2}$$

Hint: Start from Parseval's identity and argue as in the proof of Theorem 2.34.

(b) Show that u assumes the initial datum f in the following L^2 sense:

$$\lim_{t\downarrow 0} \|u(t,\cdot) - f\|_{L^2(-\pi,\pi)} = 0. \tag{3}$$

(c) Consider a function $v(t,x)$ defined in $(0,\infty)\times\mathbb{R}$ which is 2π-periodic and of class C^2 in space, and of class C^1 in time. Show that if v satisfies equations (2) and (3), then $v = u$.

Exercise 9.5.
Consider the following evolution problem with periodic boundary conditions:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u + \lambda u = 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R},\ \text{where } \lambda \geq 0 \text{ is a given constant,} \\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } 2\pi\text{-periodic } f \in C^\infty(\mathbb{R}), \\ \partial_t u(0,x) = g(x) & \text{for some given } 2\pi\text{-periodic } g \in C^\infty(\mathbb{R}). \end{cases}$$

Remark: This PDE is known as the Klein-Gordon equation. For $\lambda = 0$ it is just the wave equation.
(a) Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t) e^{ikx}$, where the $u_k(t)$ depend on λ and the Fourier coefficients of f and g.
Hint: Recall that λ is non-negative. You will get the equation for a harmonic oscillator.

(b) Show that the formal solution is in fact a true solution and is C^∞ in both variables.

(c) Show that if we just want our solution u to be in $C^2(\mathbb{R}\times\mathbb{R})$, the assumptions on f and g can be relaxed to:

$$\sum_{k\in\mathbb{Z}} \big(|k|^2 |c_k(f)| + |k||c_k(g)|\big) < \infty.$$

(d) Assume that $\lambda = 0$, i.e. we are considering the wave equation. Show that for each pair of 2π-periodic functions $\phi, \psi \in C^2(\mathbb{R})$ the function $(x,t) \mapsto \phi(x-t) + \psi(x+t)$ solves the wave equation. Explain why this is compatible with what you found in the previous points.