Exercise 10.1. ♣
Which of the following statements are true?
(a) If $f \in C^m(\mathbb{R})$ and for all derivatives up to order m, we have $f, f', \ldots, f^{(m)} \in L^1(\mathbb{R})$, then

$$\lim_{|\xi|\to\infty} |\xi|^m \hat{f}(\xi) = 0.$$

(b) Let $f \in L^1(\mathbb{R}^d)$ and let A be an invertible $d \times d$ matrix. Then we have

$$\widehat{f \circ A}(\xi) = \hat{f}(A^{-1}\xi).$$

(c) For $f, g \in \mathcal{S}(\mathbb{R})$, the Fourier transform of $h(x,y) = f(2x)g(y/2)$ is $\hat{h}(\xi,\eta) = \hat{f}(\xi/2)\hat{g}(2\eta)$.

(d) Let $\psi \in C_c^1(\mathbb{R}^d)$ with $\psi(x) \equiv 1$ in a neighbourhood of $x = 0$. Then for each $f \in L^1(\mathbb{R}^d)$:

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f(x)\psi(\epsilon x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx \quad \text{and} \quad \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\epsilon x)\,dx = 0.$$

Exercise 10.2.
Let $f \in L^1(\mathbb{R})$ be a continuous function with Fourier transform

$$\hat{f}(\xi) = \frac{\log(1+\xi^2)}{\xi^2}.$$

Compute the following:
(a) $\int_{\mathbb{R}} f(x)\,dx$,

(b) $f(0)$.

Exercise 10.3.
(a) Compute the Fourier transform of $f(x) = e^{-ax^2}$ for $a > 0$.

(b) Compute the convolution $e^{-ax^2} * e^{-bx^2}$ for $a, b > 0$ by using the Fourier transform.

Exercise 10.4.
Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Show that the sum $\sum_{n\in\mathbb{Z}} f(\sqrt{2\pi}n)$ is convergent and prove the Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f\big(\sqrt{2\pi}n\big) = \sum_{n\in\mathbb{Z}} \hat{f}\big(\sqrt{2\pi}n\big).$$

Hint: Consider the $\sqrt{2\pi}$-periodic function defined by $F(x) = \sum_{n\in\mathbb{Z}} f(x + \sqrt{2\pi}n)$.

Exercise 10.5.
The goal of this problem is to show the existence of the harmonic extension to the interior of the unit disk of a sufficiently regular function f defined on the disk's boundary.
Consider the following second order differential operators in two variables (x_1, x_2):

$$\Delta := \partial_{11} + \partial_{22} \qquad \text{and} \qquad L := \partial_{11} + \frac{1}{x_1}\partial_1 + \frac{1}{x_1^2}\partial_{22}.$$

We say that a twice differentiable function $w(x_1, x_2)$ is harmonic if $\Delta w = 0$ in its domain.
(a) Let $D := \{(x, y) : x^2 + y^2 < 1\}$ be the unit disk. Given $u\colon \overline{D} \to \mathbb{R}$, consider the function[a]

$$v(r, \theta) := u(r\cos\theta, r\sin\theta), \quad r \in [0,1],\ \theta \in \mathbb{R}. \tag{1}$$

Using the chain rule, check that

$$(\Delta u)(r\cos\theta, r\sin\theta) = Lv(r,\theta), \quad \forall\, r \in (0,1),\ \theta \in \mathbb{R}.$$

(b) Given any sufficiently regular function $F\colon \partial D \to \mathbb{R}$ consider its 2π-periodic version $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(\theta) := F(\cos\theta, \sin\theta), \quad \theta \in \mathbb{R}.$$

Show that we can find a solution $u\colon \overline{D} \setminus \{0\} \to \mathbb{R}$ of

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \{0\}, \\ u = F & \text{on } \partial D, \end{cases}$$

by instead solving

$$\begin{cases} \partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & \text{in } (0,1) \times \mathbb{R}, \\ v(r, \theta + 2\pi) = v(r, \theta) & \text{in } (0,1] \times \mathbb{R}, \\ v(1, \theta) = f(\theta) & \text{for all } \theta \in \mathbb{R}, \end{cases} \tag{2}$$

and then defining u using (1).

(c) Formally solve the system (2) by using the ansatz $v(r,\theta) := \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\theta}$. Explain why the $\{u_k(r)\}$ are not *uniquely* determined by the Fourier coefficients $\{c_k(f)\}$. Explain why they are unique if we further require that

$$\limsup_{r \downarrow 0} |u_k(r)| < \infty \quad \forall k \in \mathbb{Z}. \tag{3}$$

(d) Let $v(r,\theta)$ be the ansatz constructed in the previous subquestion by requiring the extra condition (3). Show that v is of class C^∞ for $(r,\theta) \in (0,1) \times \mathbb{R}$, as soon as $f \in L^2((-\pi,\pi))$.

(e) ★ Show that, when $f \in L^2((-\pi,\pi))$, the v you constructed with the extra condition (3) in fact corresponds to a u that is of class C^∞ in the whole open disk (including the origin!). Furthermore, this u satisfies $\Delta u = 0$ in D and meets the boundary condition in the sense that

$$\lim_{r\uparrow 1} \big\| u(r\cos(\cdot), r\sin(\cdot)) - F(\cos(\cdot), \sin(\cdot)) \big\|_{L^2((-\pi,\pi))} = 0.$$

[a] This is u in polar coordinates.