Exercise 11.1. ♣
Which of the following statements are true?
(a) If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$, then necessarily also $f \in L^2(\mathbb{R}^d)$.

(b) If $\lambda \in \mathbb{C}$ is an eigenvalue[a] of $\mathcal{F}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, then necessarily $\lambda \in \{\pm 1, \pm i\}$.

(c) The function $x \to \frac{1}{1+ix^4}$ is an element of the Schwartz class $\mathcal{S}(\mathbb{R})$.

(d) Let $f \in C^\infty(\mathbb{R})$ be a smooth function with all derivatives bounded on $\mathbb{R}$, i.e. $f^{(j)} \in L^\infty(\mathbb{R})$ for all $j \in \mathbb{N}_0$. Then

$$f\psi \in \mathcal{S}(\mathbb{R}), \quad \forall\, \psi \in \mathcal{S}(\mathbb{R}).$$

Hint: Recall the Leibniz formula for higher-order derivatives of products

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

[a] That is to say: there exists some nonzero function $v \in L^2(\mathbb{R}^d)$ such that $\mathcal{F}v = \lambda v$.

Exercise 11.2.
For $\lambda > 0$, define

$$f(x) = \begin{cases} 1 - \lambda^{-1}|x| & \text{for } |x| < \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that

$$\hat{f}(\xi) = \frac{2}{\lambda}\sqrt{\frac{2}{\pi}} \cdot \frac{\sin^2(\frac{\lambda}{2}\xi)}{\xi^2}.$$

(b) Using the Poisson summation formula in the form[a]

$$\sum_{n\in\mathbb{Z}} \hat{f}\big(\sqrt{2\pi}(\alpha+n)\big) = \sum_{n\in\mathbb{Z}} e^{-i2\pi\alpha n} f\big(\sqrt{2\pi}n\big), \quad \forall\, \alpha \in \mathbb{R},$$

and an appropriate choice of λ in the previous part, show that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}, \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Z}.$$

[a] This follows from Exercise 10.4 on the last problem set.

Exercise 11.3.
(a) Compute

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(x)}{1+x^2} e^{-i\xi x}\, dx, \quad \text{for } \xi \in \mathbb{R} \setminus \{-1, 1\}.$$

Hint: Extend the integral to a contour integral in the complex plane and apply Cauchy's integral formula.

(b) Deduce that the function $x \to \frac{x\sin(x)}{1+x^2}$ is not in $L^1(\mathbb{R})$.

Exercise 11.4.
(a) Let $f \in L^1(\mathbb{R})$ have compact support, i.e. there exists some $R > 0$ such that

$$f = 0 \quad \text{almost everywhere in } \mathbb{R} \setminus [-R, R].$$

Show that $\hat{f}$ is analytic on $\mathbb{R}$. That is, for every $\xi_0 \in \mathbb{R}$ the Taylor series of f around ξ_0 converges to f in a neighborhood of ξ_0.

(b) Let $f \in L^1(\mathbb{R})$ be a continuous function, which is not identically zero. Show that f and its Fourier transform cannot both be compactly supported.

Exercise 11.5.
Given $\phi \in \mathcal{S}(\mathbb{R})$, we consider the differential equation

$$u'(x) + u(x) = \phi(x), \quad \text{for all } x \in \mathbb{R}.$$

(a) Show that, if there is a solution $u \in \mathcal{S}(\mathbb{R})$, then it is the unique solution within the class of Schwartz functions.

(b) Using the Fourier transform and inverse Fourier transform, show that there is indeed a solution $u \in \mathcal{S}(\mathbb{R})$ to the differential equation.

(c) Solve the differential equation again, this time with classical methods (multiply by e^t etc..).

(d) Check that the two results you found are indeed the same.