Exercise 12.1. ♣

Which of the following statements are true?

(a) Define $f_n(\xi) = e^{-in\xi}\frac{\sin(\xi)}{\sqrt{\pi}\xi}$. Then $\{f_n \,|\, n \in \mathbb{Z} \text{ even}\}$ is an orthonormal system in $L^2(\mathbb{R})$.

Hint: Think in terms of the Fourier transform.

(b) Let $f \in S(\mathbb{R}^d)$ be a function whose Fourier transform is supported in the ball of radius $\epsilon > 0$, i.e. $\operatorname{supp}(\hat{f}) \subset B_\epsilon$. Then we must have

$$\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, dx \geq \frac{d^2}{4}\epsilon^{-2}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

(c) Let $I \subset \mathbb{R}$ be an open interval and $h \in L^\infty(I)$ with h not identically zero. Then the operator

$$T : L^2(I) \to L^2(I), \quad Tf(x) = h(x)f(x)$$

is compact.

(d) Let V be a finite dimensional inner product space and H a Hilbert space. Then any linear operator $T : V \to H$ is compact.

(e) Let (X, d) be a metric space and let $x \in X$. Assume $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ must possess a sub-subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ with $x_{n_{k_j}} \to x$ as $j \to \infty$. Then $x_n \to x$ as $n \to \infty$.

Exercise 12.2.

Consider the Heisenberg inequality on $\mathbb{R}$:

$$\|xf(x)\|_{L^2(\mathbb{R})} \cdot \|\xi\hat{f}(\xi)\|_{L^2(\mathbb{R})} \geq \frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}, \quad \forall f \in \mathcal{S}(\mathbb{R}).$$

Show that equality holds if and only if $f(x) = Ce^{-\lambda x^2}$ for some $C \in \mathbb{R}$ and $\lambda > 0$.

Hint: When does equality hold for the Cauchy-Schwarz inequality?

Exercise 12.3.

Let H be a Hilbert space. Denote the set of compact operators from H to H by $\mathcal{K}(H)$ and the set of bounded operators by $\mathcal{B}(H)$.

(a) Show that $\mathcal{K}(H)$ is a linear subspace of $\mathcal{B}(H)$.

(b) Show that $\mathcal{K}(H)$ is a two-sided ideal in $\mathcal{B}(H)$ with respect to composition, that is for any $T \in \mathcal{K}(H)$ and $S \in \mathcal{B}(H)$, we have $ST \in \mathcal{K}(H)$ and $TS \in \mathcal{K}(H)$.

Exercise 12.4.
Let $(X, \|\cdot\|)$ be an infinite dimensional normed vector space.
(a) Let $Y \subset X$ be a proper closed linear subspace, i.e. $Y \neq X$. Show that there exists $x \in X$ satisfying $\|x\| = 1$ and $\|x - y\| \geq \frac{1}{2}$ for all $y \in Y$.
Hint: Argue that we can find $x_0 \in X$ with $\alpha := \inf_{y \in Y} \|x_0 - y\| > 0$ and $y_0 \in Y$ with $\alpha \leq \|x_0 - y_0\| \leq 2\alpha$. Then consider $x = \|x_0 - y_0\|^{-1}(x_0 - y_0)$.

(b) Show that the closed unit ball $\overline{B_1} \subset X$ is not compact.
Hint: Use the first part of this exercise to construct a sequence $(x_n)_{n\in\mathbb{N}}$ contained in $\overline{B_1}$ and satisfying $\|x_n - x_m\| \geq \frac{1}{2}$ for all $m \neq n$.

Exercise 12.5.
Let $f \in L^2(\mathbb{R}^d)$ be a function whose Fourier transform decays at infinity as a negative power, i.e. for some $\alpha, M > 0$ we have

$$|\hat{f}(\xi)| \leq M|\xi|^{-\alpha} \quad \text{for all } |\xi| \geq 1.$$

The goal of this problem is to show that in fact $f \in C^k(\mathbb{R}^d)$ for all non-negative integers $k < \alpha - d$. (More precisely, f has a representative in $C^k(\mathbb{R}^d)$.)
(a) For each $R > 1$, consider the function

$$f_R(x) := (2\pi)^{-d/2} \int_{B_R} \hat{f}(\xi) e^{i\xi x} \, d\xi,$$

compute $\hat{f}_R$ and show that $f_R \to f$ in $L^2(\mathbb{R}^d)$ as $R \to \infty$.

(b) Show that $f_R \in C^\infty(\mathbb{R}^d)$ for any R, but in general $f_R \notin \mathcal{S}(\mathbb{R}^d)$.

(c) Assume now that $\alpha > d$. Using the decay assumption on $\hat{f}$, show that $(f_R)_{R>0}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^d)$. Conclude that $f \in C(\mathbb{R}^d)$ (up to re-definition on a zero measure set).

(d) Applying the same argument to $\partial_{x_j} f_R$, show inductively that $f \in C^k(\mathbb{R}^d)$ whenever $\alpha > d + k$.