Exercise 13.1. ♣
Which of the following statements are true?
(a) If $T : H \to H$ is a compact operator on a Hilbert space, then $0 \in \mathrm{EV}(T)$.

(b) If $T : H \to H$ is a compact operator on a Hilbert space, then $0 \in \sigma(T)$.

(c) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then the operator

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad T\big((x_n)_{n\in\mathbb{N}}\big) = (\alpha_n x_n)_{n\in\mathbb{N}}$$

is compact.

(d) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then the spectrum of the operator

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad T\big((x_n)_{n\in\mathbb{N}}\big) = (\alpha_n x_n)_{n\in\mathbb{N}}$$

satisfies $\sigma(T) = \{\alpha_n\}_{n\in\mathbb{N}} \cup \{0\}$.

(e) Let H be a Hilbert space and let $v_1, \dots v_n, w_1, \dots, w_n \in H$. Then $T(x) = \sum_{k=1}^{n} \langle x,\, v_k\rangle\, w_k$ defines a bounded linear operator $T : H \to H$.

Exercise 13.2.
Consider the map $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$T((x_k)_{k\in\mathbb{N}}) = \left(\frac{x_k}{k}\right)_{k\in\mathbb{N}}.$$

(a) Show that T is a continuous linear operator and determine its norm.

(b) Show that T is the limit (with respect to the operator norm) of a sequence of finite rank operators. Is T compact?

(c) Determine the set of eigenvalues and the spectrum of T.

Exercise 13.3.
Let $\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}^+ = \{q \in \mathbb{Q},\, q > 0\}$, i.e. $n \in \mathbb{N} \to r_n \in \mathbb{Q}^+$ is a bijection. Let H be a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ a Hilbert basis in H. Define the linear operator

$$T : H \to H, \quad T(x) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{r_n} \langle x,\, e_n\rangle\, e_n.$$

(a) Show that T is a bounded operator and compute $\|T\|$.

(b) Determine the set of eigenvalues $\mathrm{EV}(T)$ and the spectrum $\sigma(T)$ of T.

(c) Is T a compact operator?

Exercise 13.4.
In this exercise you will construct a compact operator, which is in some sense a right inverse for the Laplace operator $-\frac{d^2}{dx^2}$ on $[0,\pi]$.
(a) Let $f \in L^2([0,\pi];\mathbb{R})$ and consider the following differential equation with boundary conditions:

$$\begin{cases} -u'' = f, \\ u(0) = u(\pi) = 0. \end{cases} \tag{2}$$

Recall from Exercise 7.3 that $B = \{\sqrt{2/\pi}\sin(kx)\}_{k\in\mathbb{N}}$ is a Hilbert basis for $L^2([0,\pi];\mathbb{R})$. Write both u and f as Fourier series of sines on $[0,\pi]$ and formally find a solution u of (2).

(b) Show that for $f \in L^2([0,\pi];\mathbb{R})$, the formal solution satisfies $u \in L^2([0,\pi];\mathbb{R}) \cap C([0,\pi])$ and the boundary condition $u(0) = u(\pi) = 0$ is satisfied.

(c) Prove that the map assigning f to the formal solution u of (2)

$$T\colon L^2([0,\pi];\mathbb{R}) \to L^2([0,\pi];\mathbb{R}), \quad T(f) = u$$

is a continuous linear operator, which is moreover self-adjoint and compact.

(d) Show that the set of functions

$$\big\{u \in C^2([0,\pi];\mathbb{R}),\, u(0) = u(\pi) = 0\big\}$$

is contained in the image of T. What is a sufficient condition on the Fourier coefficients of f to make sure that $u = T(f)$ is C^2, i.e. that it is a classical solution of (2)?

(e) We have shown that the map

$$P : \mathrm{Im}(T) \to L^2([0,\pi];\mathbb{R}), \quad P(u) = f, \quad \text{where } u = T(f),$$

is an extension of the operator $-\frac{d^2}{dx^2}$ from the subspace

$$\big\{u \in C^2([0,\pi];\mathbb{R}),\, u(0) = u(\pi) = 0\big\} \subset L^2([0,\pi];\mathbb{R})$$

to the larger subspace $\mathrm{Im}(T) \subset L^2([0,\pi];\mathbb{R})$. By the Spectral Theorem for compact operators, we know that there is a Hilbert basis consisting of eigenfunctions of T. Show that the same eigenfunctions are eigenfunctions of P.

Remark: We have found a Hilbert basis of eigenfunctions for the Laplace operator $-\frac{d^2}{dx^2}$ with Dirichlet boundary conditions, i.e. an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}} \subset L^2([0,\pi];\mathbb{R})$ and eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\begin{cases} -\varphi_n'' = \lambda_n e_n, \\ \varphi_n(0) = \varphi_n(\pi) = 0. \end{cases}$$

Of course we've been working with said eigenfunctions since the beginning of this exercise, what are they? A similar strategy can be used to find a Hilbert basis of eigenfunctions for the Laplace operator with Dirichlet boundary conditions on more complicated domains.