Exercise 1.1. ♣
Which of the following statements are true? There may be more than one true statement.

(a) If $f \in L^1(\mathbb{R}^n)$ then necessarily $f \in L^2(\mathbb{R}^n)$? ✗

(b) If $f \in \ell^1(\mathbb{N})$ then necessarily $f \in \ell^2(\mathbb{N})$? ✓

(c) Let $(f_n)_{n\in\mathbb{N}}$ be a collection of non-negative measurable functions. Is it true that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)\, dx\,?$$

✓

(d) Is it possible to find a sequence of functions $\{f_k\} \subset L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |f_k(x) - 1|^2\, dx \to 0 \text{ as } k \to \infty\,?$$

✗

(e) A Cauchy sequence (say, in a metric space) can have at most one limit point? ✓

(f) The interval $(0, 1) \subset \mathbb{R}$ is complete? ✗

Solution:

(a) No, there is no inclusion between these spaces. For example, in $\mathbb{R}^n$ take $f(x) = 1/(1 + |x|^n)$ and $g(x) = |x|^{-n/2}e^{-|x|}$. Then $f \in L^2(\mathbb{R}^n) \setminus L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$.

(b) Yes. Since $f \in \ell^1(\mathbb{N})$ is summable it is necessarily bounded, i.e. $|f(j)| \le C$ for all $j \in \mathbb{N}$ and some $C > 0$. Hence $\sum_{j\in\mathbb{N}} f(j)^2 \le C \sum_{j\in\mathbb{N}} f(j) < \infty$.

(c) Yes, as a direct consequence of the Monotone Convergence Theorem applied to the partial sums $g_N(x) = \sum_{n=1}^{N} f_n(x)$.

(d) Suppose one could construct such a sequence. Then, by the triangle inequality

$$\|f\|_{L^2} \le \|f_k - f\|_{L^2} + \|f_k\|_{L^2} \le 1 + \|f_k\|_{L^2}$$

for k sufficiently large. Since $f_k \in L^2$, this would imply that $f(x) = 1 \in L^2$, but $1 \notin L^2(\mathbb{R})$ and we get a contradiction.

(e) Yes. Assume that a, b are both limit points of a Cauchy sequence $\{x_n\}$. Take converging subsequences $x_{k_n} \to a$ and $x_{j_n} \to b$. The triangle inequality gives

$$d(a, b) \le d(a, x_{k_n}) + d(x_{k_n}, x_{j_n}) + d(x_{j_n}, b).$$

Now for any $\varepsilon > 0$, we can choose n large enough, so that the first and second terms are smaller than ε by the convergence of the respective subsequence, and the middle term is smaller than ε by the Cauchy property of the sequence. Thus, $d(a, b) = 0$, i.e. $a = b$.

(f) No, the sequence $x_n := 2^{-n}$ is Cauchy, but its limit point in $\mathbb{R}$ (i.e., zero) does not lie in $(0, 1)$.

Exercise 1.2.
Consider the sequence of functions $f_n\colon (0,\infty) \to \mathbb{R}$ defined by

$$f_n(x) = \frac{\arctan(x)}{x}\chi_{(0,n)}(x).$$

Determine the pointwise limit f and discuss the convergence $f_n \to f$ in $L^2\big((0,\infty)\big)$. Is the limit also in $L^1\big((0,\infty)\big)$? What can we deduce about the completeness of the space L^1 with respect to the norm $\|\cdot\|_{L^2}$?

Solution: The pointwise limit is clearly

$$f(x) = \frac{\arctan(x)}{x}.$$

We note that $f_n \in L^2\big((0,\infty)\big)$ for every n, since the f_n are compactly supported bounded functions (notice that $\lim_{x\to 0}\arctan(x)/x = 1$). Moreover, the sequence f_n converges to f in $L^2\big((0,\infty)\big)$. Indeed

$$\|f-f_n\|_{L^2}^2 = \int_0^\infty |f(x)-f_n(x)|^2\,dx = \int_n^\infty \left|\frac{\arctan(x)}{x}\right|^2 dx \leq \Big(\frac{\pi}{2}\Big)^2\int_n^\infty \frac{1}{x^2}\,dx = \Big(\frac{\pi}{2}\Big)^2\frac{1}{n}.$$

Thus, $\|f-f_n\|_{L^2} \to 0$ as $n\to\infty$.

Now, observe that $f_n \in L^1\big((0,\infty)\big)$ for every n, but the limit f is not in L^1. Indeed, for $x \geq 1$ we have $\frac{\arctan(x)}{x} \geq \frac{\pi}{4}\frac{1}{x}$, which is not integrable on $(1,\infty)$. Since the sequence f_n is Cauchy with respect to the L^2 norm, this shows that L^1 is not complete with respect to the L^2 norm.

Exercise 1.3.
Recall that the Dominated Convergence Theorem implies that a collection of measurable functions $f_n\colon \mathbb{R}\to\mathbb{C}$, satisfying $|f_n| \leq g$ for some $g \in L^1(\mathbb{R})$, also satisfies

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}}\Big(\lim_{n\to\infty} f_n(x)\Big)\,dx$$

whenever the pointwise limit $\lim_{n\to\infty} f_n(x)$ exists a.e. Show, via a counterexample, that the hypothesis $|f_n| \leq g \in L^1(\mathbb{R})$ is necessary.

Hint: Can you think of an example in which the statement fails? For instance, a sequence of functions f_n with constant integral (>0) but with poitwise limit 0?

Solution: For instance the functions $f_n(x) := n\cdot\chi_{(0,1/n)}(x)$ provide a counterexample. Indeed, for any $x\in\mathbb{R}$ the value $f_n(x)$ is eventually 0, i.e. the f_n converge pointwise to 0. However, for any $n\in\mathbb{N}$

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_0^{\frac{1}{n}} n\,dx = 1.$$

Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx = 1 \neq 0 = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\,dx.$$

Note as an aside that this implies any function g dominating the sequence cannot be integrable. Indeed, the lowest dominant is

$$h(x) := \sup_{n\in\mathbb{N}} f_n(x),$$

which is bounded by any dominant g, and satisfies

$$\int_{\mathbb{R}} g\,dx \geq \int_{\mathbb{R}} h\,dx = \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} n\,dx = \sum_{n=1}^{\infty}\frac{1}{n+1} = \infty.$$

Exercise 1.4.
Consider the space $C\big([0,1]\big)$ of continuous functions on $[0,1]$ equipped with the norm

$$\|f\|_\infty = \max_{x\in[0,1]}|f(x)|.$$

Show that this is not an inner product space, i.e. there is no inner product on $C\big([0,1]\big)$ which induces this norm.

Hint: Recall the parallelogram law.

Solution: For any norm induced by an inner product the parallelogram law must hold, that is

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2, \quad \forall f,g.$$

We show that the parallelogram law with respect to the given norm does not hold for two elements in $C\big([0,1]\big)$. Define

$$f,g:[0,1]\to\mathbb{R}, \qquad f(x)=x, \quad g(x)=1-x.$$

Then $f,g\in C\big([0,1]\big)$ and we calculate

$$\|f\|_\infty = \max_{x\in[0,1]}|x| = 1, \quad \|g\|_\infty = \max_{x\in[0,1]}|1-x| = 1.$$
$$\|f+g\|_\infty = \max_{x\in[0,1]} 1 = 1, \quad \|f-g\|_\infty = \max_{x\in[0,1]}|2x-1| = 1.$$

Thus,

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 2 \neq 4 = 2\big(\|f\|_\infty^2 + \|g\|_\infty^2\big).$$

Exercise 1.5.
Consider the space $\ell^p(\mathbb{N})$ of p-summable sequences (for $p \geq 1$) with the norm

$$\|x\|_p = \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{\frac{1}{p}}.$$

Show that $\ell^p(\mathbb{N})$ with $p \neq 2$ is not an inner product space.

Solution: Once again, we exhibit two elements $x, y \in \ell^p(\mathbb{N})$ for which the parallelogram law does not hold. Define

$$x = (1, 1, 0, 0, \cdots), \quad y = (1, -1, 0, 0, \cdots).$$

Then

$$\|x\|_p = \|y\|_p = 2^{\frac{1}{p}}, \qquad \|x + y\|_p = \|x - y\|_p = 2.$$

Thus,

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 8 \neq 4 \cdot 2^{\frac{2}{p}} = 2\big(\|x\|_p^2 + \|y\|_p^2\big),$$

since $p \neq 2$.