Exercise 2.1.

(a) Let $V := \mathrm{M}_{n\times n}(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries and define the *Fobenius product* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ as

$$\langle A, B\rangle := \mathrm{Tr}(AB^\dagger) = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}$$

where Tr denotes the trace and $B^\dagger$ is the Hermitian transpose of B, obtained by transposition and complex conjugation of the entries: $B^\dagger = \overline{B^T}$. Show that $(V, \langle\cdot,\cdot\rangle)$ is an inner-product space.

Hint: first observe that $\mathrm{Tr}(A) = \overline{\mathrm{Tr}(A^\dagger)}$.

(b) Consider n inner-product spaces $(V_1, \langle\cdot,\cdot\rangle_1), \dots, (V_n, \langle\cdot,\cdot\rangle_n)$. Show that $(V, \langle\cdot,\cdot\rangle)$, where $V = V_1 \times \dots \times V_n$ and

$$\langle (v_1, \dots, v_n), (w_1, \dots, w_n)\rangle := \sum_{i=1}^{n} \langle v_i, w_i\rangle_i,$$

is an inner product space.

(c) Let $W := M_{n\times n}(L^2(\mathbb{R}, \mathbb{C}))$ be the space of $n \times n$ matrices whose entries are square integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Which product would make W an inner product space?

Hint: observe that W is a "composition" of two inner product spaces.

Solution:

(a) Note that $\mathrm{M}_{n\times n}(\mathbb{C})$ is a vector space. We need to check that $\langle\cdot,\cdot\rangle$ satisfies the three axioms of inner product space: conjugate symmetry, linearity in the first argument and positive definiteness. First observe that the property $\mathrm{Tr}(A) = \overline{\mathrm{Tr}(A^\dagger)}$ follows directly from the fact that the trace is invariant by transposition. Notice also that Hermitian transposition is an involution, i.e. $(B^\dagger)^\dagger = B$. Then

$$\langle A, B\rangle = \mathrm{Tr}(AB^\dagger) = \overline{\mathrm{Tr}((AB^\dagger)^\dagger)} = \overline{\mathrm{Tr}((B^\dagger)^\dagger A^\dagger)} = \overline{\mathrm{Tr}(BA^\dagger)} = \overline{\langle B, A\rangle}$$

and hence conjugate symmetry holds. Linearity in the first argument follows trivially by linearity of the trace. Finally, if $A \neq 0$,

$$\mathrm{Tr}(AA^\dagger) = \sum_{i,j=1}^{n} a_{ij}\overline{a_{ij}} = \sum_{i,j=1}^{n} |a_{ij}|^2 > 0.$$

which proves positive definiteness.

(b) Note that the Cartesian product of vector spaces is a vector space. Once again, we check the three axioms for an inner product. These follow from the corresponding axioms for the inner product spaces $(V_i, \langle\cdot,\cdot\rangle_i)$. Denote $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$. We readily check that

$$\langle v, w\rangle = \sum_i \langle v_i, w_i\rangle_i = \sum_i \overline{\langle w_i, v_i\rangle}_i = \overline{\langle w, v\rangle}$$

and

$$\langle \alpha v + \beta u, w\rangle = \sum_i \langle \alpha v_i + \beta u_i, w_i\rangle_i = \sum_i \alpha\langle v_i, w_i\rangle_i + \beta\langle u_i, w_i\rangle_i = \alpha\langle v, w\rangle + \beta\langle u, w\rangle$$

which prove the first two axioms. For positive definiteness, note that if $v \neq 0$ then $v_k \neq 0$ for some $k \in \{1, \dots, n\}$. Thus

$$\langle v, v\rangle = \sum_{i=1}^{n} \langle v_i, v_i\rangle_i \geq \langle v_k, v_k\rangle_k > 0.$$

(c) Note that $M_{n\times n}(L^2(\mathbb{R}, \mathbb{C}))$ is a vector space under component-wise addition and scalar multiplication. Let $F = (f_{ij})_{1\leq i,j\leq n}$ and $G = (g_{ij})_{1\leq i,j\leq n}$ be two matrices of square integrable functions. Define

$$\langle F, G\rangle = \sum_{i,j=1}^{n} \int_{\mathbb{R}} f_{ij}(x)\overline{g_{ij}(x)}dx.$$

This product is a natural choice since it's the "composition" of the Frobenius product (defined above) and the L^2 inner product. It's straightforward to check that all the axioms of an inner product space hold.

Remark: this "composition trick" was implicitly used also in part 2. Indeed the inner product there is a composition of the ones of the respective V_i and the one in $\mathbb{R}^n$, given by $a \cdot b = \sum_i a_i b_i$.

Exercise 2.2.
An inner product space $(V, \langle\cdot,\cdot\rangle)$ is also a metric space under the norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$, hence it has a natural topology (induced by the metric $d(v, w) = \|v - w\|$). Prove that the vector space operations $+ : V \times V \to V$ and $\cdot : \mathbb{C} \times V \to V$ and the inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ are continuous, where $V \times V$ and $\mathbb{C} \times V$ are endowed with the natural product topologies. Recall that the topology in $V \times V$ is the one induced by the norm

$$\|(v_1, v_2)\|_{V\times V} := \|v_1\| + \|v_2\|.$$

Similarly, the norm (thus the metric and the topology) on $\mathbb{C} \times V$ is given by

$$\|(\alpha, v)\|_{\mathbb{C}\times V} := |\alpha| + \|v\|.$$

Solution: Recall that a map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is continuous iff for each $x_1 \in X$ and each $\epsilon > 0$ there is $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \epsilon$.

The continuity of the sum is follows from the triangle inequality. For $(v_1, v_2), (w_1, w_2) \in V \times V$, we have

$$\|(v_1 + v_2) - (w_1 + w_2)\| \leq \|v_1 - w_1\| + \|v_2 - w_2\| = \|(v_1, v_2) - (w_1, w_2)\|_{V\times V}.$$

So if (v_1, v_2) and (w_1, w_2) are ϵ-close in $V \times V$ (i.e., the right hand side is smaller than ϵ) then $v_1 + v_2$ is ϵ-close to $w_1 + w_2$ in V.

The continuity of scalar multiplication follows from homogeneity and the triangle inequality. Indeed, for $(\alpha, v), (\beta, w) \in \mathbb{C} \times V$, we have

$$\|\alpha v - \beta w\| = \|\alpha v - \beta v + \beta v - \beta w\| \leq \|(\alpha - \beta)v\| + \|\beta(v - w)\| \leq |\alpha - \beta|\|v\| + |\beta|\|v - w\|$$

Now $|\beta| \leq |\alpha| + |\beta - \alpha|$, so

$$\begin{aligned}\|\alpha v - \beta w\| &\leq \|v\||\alpha - \beta| + |\alpha|\|v - w\| + |\alpha - \beta|\|v - w\| \\ &\leq \max\{|\alpha|, \|v\|\}\|(\alpha, v) - (\beta, w)\|_{\mathbb{C}\times V} + \|(\alpha, v) - (\beta, w)\|^2_{\mathbb{C}\times V}\end{aligned}$$

Thus, given $(\alpha, v) \in \mathbb{C} \times V$ and $\epsilon > 0$, we can choose $\|(\alpha, v) - (\beta, w)\|$ small enough so that $\|\alpha v - \beta w\| < \epsilon$

The continuity of the inner product follows by applying the Cauchy and triangle inequalities. For $(v_1, v_2), (w_1, w_2) \in V \times V$, we find

$$\begin{aligned}|\langle v_1, v_2\rangle - \langle w_1, w_2\rangle| &= |\langle v_1, v_2\rangle - \langle w_1, v_2\rangle + \langle w_1, v_2\rangle - \langle w_1, w_2\rangle| \leq |\langle v_1 - w_1, v_2\rangle| + |\langle w_1, v_2 - w_2\rangle| \\ &\leq \|v_1 - w_1\|\|v_2\| + \|w_1\|\|v_2 - w_2\| \\ &\leq \|v_1 - w_1\|\|v_2\| + \|v_1\|\|v_2 - w_2\| + \|v_1 - w_1\|\|v_2 - w_2\| \\ &\leq \max\{|v_1|, \|v_2\|\}\|(v_1, v_2) - (w_1, w_2)\|_{V\times V} + \|(v_1, v_2) - (w_1, w_2)\|^2_{V\times V},\end{aligned}$$

Thus, given $(v_1, v_2) \in V \times V$ and $\epsilon > 0$, we can choose $\|(v_1, v_2) - (w_1, w_2)\|$ small enough so that $|\langle v_1, v_2\rangle - \langle w_1, w_2\rangle| < \epsilon$.

Exercise 2.3.
Show that $\ell^p(\mathbb{N})$ for $0 < p < 1$ is a vector space but not a normed space, that is $\left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}}$ does not define a norm on $\ell^p(\mathbb{N})$.

Solution: We first remark that

$$\ell^p(\mathbb{N}) = \left\{(x_n)_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\right\}$$

is a vector space. It is immediate that $x \in \ell^p(\mathbb{N}) \implies \alpha x \in \ell^p(\mathbb{N})$ for $\alpha \in \mathbb{C}$. To see that $\ell^p(\mathbb{N})$ is closed under taking sums, note that

$$(1 + t)^p \leq 1 + t^p, \quad \forall t \geq 0.$$

Indeed, the two sides agree at $t = 0$, and the derivatives satisfy

$$\partial_t(1 + t)^p = p(1 + t)^{p-1} \leq pt^{p-1} = \partial_t(1 + t^p), \quad \forall t \geq 0,$$

since $p - 1 < 0$. It now follows that

$$(a + b)^p \leq a^p + b^p, \quad \forall a, b \geq 0,\, 0 < p < 1.$$

Thus, for $x, y \in \ell^p(\mathbb{N})$, we have

$$|x_n + y_n|^p \leq (|x_n| + |y_n|)^p \leq |x_n|^p + |y_n|^p, \quad \forall n \in \mathbb{N},$$

and finally

$$\sum_{n\in\mathbb{N}} |x_n + y_n|^p \leq \sum_{n\in\mathbb{N}} |x_n|^p + \sum_{n\in\mathbb{N}} |y_n|^p,$$

i.e. $x, y \in \ell^p(\mathbb{N}) \implies x + y \in \ell^p(\mathbb{N})$.

The function $x \in \ell^p(\mathbb{N}) \to h(x) = \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}}$ is not a norm, because the triangle inequality is not satisfied. Indeed, for

$$x = (1, 0, 0, \cdots), \qquad y = (0, 1, 0, \cdots)$$

we find $h(x) = h(y) = 1$ and $h(x + y) = 2^{\frac{1}{p}}$. Since $0 < p < 1$, we thus have

$$h(x + y) > h(x) + h(y).$$

Exercise 2.4. ♣

In the normed space $(L^2((0,1)), \|\cdot\|_{L^2})$, consider the subset

$$X = \left\{f \in L^2((0,1)) : \int_0^1 f\,dx = 1\right\}.$$

Which of the following statements are true?

- ☐ X is not well-defined.
- ☐ X is well-defined, open and convex.
- ☐ X is well-defined, closed, convex but not a linear subspace.
- ☐ X is well-defined, closed and a linear subspace.

Solution: X is well-defined, closed and convex, but not a linear subspace. It's is well-defined since $L^2((0,1)) \subset L^1((0,1))$. It can't be a subspace since $0 \notin X$. It's closed because if $u_k \in X$ and $u_k \to u$ in L^2 then

$$\left|\int_0^1 u_k\,dx - \int_0^1 u\,dx\right| \leq \int_0^1 |u_k - u| = \|u - u_k\|_{L^1} \leq \|u_k - u\|_{L^2} \to 0.$$

Thus, $\int_0^1 u_k\,dx \to \int_0^1 u\,dx$ and since $\int_0^1 u_k\,dx = 1$ for all k, we find $\int_0^1 u\,dx = 1$. This proves that $u \in X$, hence X contains its accumulation points, i.e. is closed. Since $X \neq L^2((0,1))$, X is closed and $L^2((0,1))$ is connected, X cannot be open; alternatively you can show that the complement of X is not closed by taking e.g. $f_k(x) = 1 + 2^{-k}$.

Convexity is immediately checked by linearity of the integral:

$$\begin{aligned} u, v \in X,\, t \in [0,1] &\implies \int_0^1 (tu + (1-t)v)\,dx = t\int_0^1 u\,dx + (1-t)\int_0^1 v\,dx = t + 1 - t = 1 \\ &\implies tu + (1-t)v \in X. \end{aligned}$$

Exercise 2.5.
In the following normed spaces, determine whether the given subsets X are well-defined, open, closed, linear subspaces and/or convex.

(a) In the normed space $(C([0,1]), \|\cdot\|_{L^\infty})$, the subset X of nowhere vanishing functions.

(b) In the normed space $(C([0,1]), \|\cdot\|_{L^2})$, the subset X of nowhere vanishing functions.

(c) In the normed space $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$, the subset

$$X = \big\{f \in L^2(\mathbb{R}) : f(x) = f(-x) \text{ for a.e. } x \in \mathbb{R}\big\}.$$

Hint: It's useful to recall that if $u_k \to u$ in L^2 then, up to picking a subsequence, there is a null measure set N such that $u_k(x) \to u(x)$ for all $x \notin N$.

(d) (★) In the normed space $(L^2((0,1)), \|\cdot\|_{L^2})$, the subset

$$X = \Big\{f \in L^2((0,1)) : f \geq 0 \text{ a.e. and } \int_0^1 \frac{2f}{1+f}\,dx \geq 1\Big\}.$$

Hint: observe that the map $s \mapsto 2s/(1+s)$ is concave for $s \geq 0$.

Solution:

(a) X is well-defined and open, but neither closed nor convex (and hence not a linear subspace). We show openness: if $u \in X$ then $\delta := \min_{x\in[0,1]} |u(x)|$ is strictly positive, so for any other $v \in C([0,1])$ with $\|u-v\|_{L^\infty} < \delta/2$, we find

$$|v(x)| \geq |u(x)| - |v(x) - u(x)| \geq |u(x)| - \|u-v\|_{L^\infty} \geq \delta - \frac{\delta}{2} > 0, \quad \forall x \in [0,1],$$

so $v \in X$. It is not closed, since the functions $f_k(x) = 2^{-k}$ belong to X but their limit does not, and X is not convex since the constant functions $1 \in X$ and $-1 \in X$, but $\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0 \notin X$.

(b) X is well-defined, but not open nor closed nor convex. It is not closed nor convex by the same examples as above. To see that X is not open, take the constant function $1 \in X$. Define for $\epsilon > 0$:

$$f_\epsilon(x) = \begin{cases} \epsilon^{-1}x, & x \in [0,\epsilon], \\ 1, & x \in [\epsilon, 1]. \end{cases}$$

Note that $f_\epsilon(x) \notin X$, since $f_\epsilon(0) = 0$. We estimate

$$\int_0^1 (1 - f_\epsilon(x))^2\,dx = \int_0^\epsilon \Big(1 - \frac{x}{\epsilon}\Big)^2\,dx \leq \int_0^\epsilon 1\,dx = \epsilon.$$

We see that $\|1 - f_\epsilon\|_{L^2} \leq \sqrt{\epsilon}$, showing that every open ball around the function $1 \in X$ contains an element outside of X. In fact, for any element $u \in X$, a similar argument using the functions $u_\epsilon(x) = \min\{\epsilon^{-1}x, u(x)\}$ even shows that no open ball is contained in X.

(c) X is well-defined, closed and a linear subspace (hence also convex). Recall that $L^2(\mathbb{R})$ is defined as a quotient space (where functions agreeing almost everywhere are identified), so there is something to check for well-definedness. If $u(x) = v(x)$ for almost every x, then also $u(-x) = v(-x)$ for almost every x. This is because if $A \subset \mathbb{R}$ has full measure then also $-A$ has full measure and so does $A \cap (-A)$. Thus, the property $u(x) = u(-x)$ a.e. is well-defined on $L^2(\mathbb{R})$.

To show that X is closed, let $u_k \to u$ in $L^2(\mathbb{R})$. Then, up to taking a sub-sequence, there is a null set $N \subset \mathbb{R}$ such that

$$u_k(x) \to u(x) \text{ and } u_k(-x) \to u(-x) \text{ for all } x \in \mathbb{R} \setminus N.$$

Thus, $u(x) - u(-x) = \lim_{k\to\infty}\big(u_k(x) - u_k(-x)\big) = 0$ for almost every $x \in \mathbb{R}$, so $u \in X$.

The fact that X is closed under linear combinations is immediate to check, let us refresh the full argument which you probably have seen in Analysis III: if

$$u(x) = u(-x) \text{ for all } x \in \mathbb{R} \setminus N_u \text{ and } v(x) = v(-x) \text{ for all } x \in \mathbb{R} \setminus N_v,$$

with $|N_u| = |N_v| = 0$, then $|N_u \cup N_v| = 0$ and for all $x \in \mathbb{R} \setminus (N_u \cup N_v)$ we have

$$\alpha u(x) + \beta v(x) = \alpha u(-x) + \beta v(-x).$$

(d) X is well defined, closed and convex. Well-defined because for all $u \in L^2, u \geq 0$, we have

$$\int_0^1 \frac{2u(x)}{1+u(x)}\,dx \leq \int_0^1 2\,dx = 2,$$

since $2s/(1+s) \in [0,2)$ for all $s \in [0,\infty)$.

X is convex because the function $\psi\colon s \mapsto \frac{2s}{1+s}$ is concave for $s \in [0,\infty)$. So if $u, v \in X$ and $t \in [0,1]$ we have for almost every x

$$\begin{aligned}\frac{tu(x) + (1-t)v(x)}{1 + tu(x) + (1-t)v(x)} &= \psi(tu(x) + (1-t)v(x)) \\ &\geq t\psi(u(x)) + (1-t)\psi(v(x)) = t\frac{u(x)}{1+u(x)} + (1-t)\frac{v(x)}{1+v(x)}.\end{aligned}$$

Integrating both sides of this inequality we find

$$\int_0^1 2\frac{tu(x) + (1-t)v(x)}{1 + tu(x) + (1-t)v(x)}\,dx \geq t\int_0^1 \frac{2u(x)}{1+u(x)}\,dx + (1-t)\int_0^1 \frac{2v(x)}{1+v(x)}\,dx \geq t\cdot 1 + (1-t)\cdot 1 = 1,$$

which shows that $tu + (1-t)v \in X$.

To check that X is closed, we pick a sequence $u_k \in X$ with $u_k \to u$ in L^2. We want to show that also $u \in X$. We find a subsequence (which we don't re-label) such that also $u_k(x) \to u(x)$ for all $x \in (0,1) \setminus N$ with $|N| = 0$. Since the u_k were nonnegative we find that also $u \geq 0$ a.e. It remains to show that $\int_0^1 \frac{u}{1+u}\,dx \geq 1$. To do so we invoke the dominated convergence theorem, i.e. since the integrands are bounded by a common function

$$\left|\frac{2u_k}{1+u_k}\right| \leq 2 \text{ uniformly in } k,$$

we can exchange pointwise limit and integral and find

$$\int_0^1 \frac{2u}{1+u} = \lim_{k\to\infty}\int_0^1 \frac{2u_k}{1+u_k} \geq 1.$$