Exercise 3.1. ♣
Which of the following pairs (vector space, bilinear form) are Hilbert spaces?
(a) $V := L^2(\mathbb{R};\mathbb{C})$ and $\langle u, v\rangle := \int_{\mathbb{R}} u(t)\bar{v}(t)\,\frac{dt}{1+t^2}$.
(b) $V := \{\text{real polynomials of degree at most } N\}$ and $\langle p, q\rangle := p(\frac{d}{dx})|_{x=0}\, q$.
Hint: If $p(X)$ is a polynomial, then $p(\frac{d}{dx})|_{x=0}$ is the differential operator obtained by replacing X with $\frac{d}{dx}$ and then evaluating at $x = 0$. Example: if $p(X) = X^2 + 3$ then $p(\frac{d}{dx})|_{x=0}q = q''(0) + 3q(0)$. Observe that $\left(\frac{d}{dx}\right)^j_{x=0} x^k = \delta^{kj}\, k!$.
(c) $V := L^1\big((0, 1);\mathbb{R}\big)$ and $\langle u, v\rangle := \int_0^1 u(x)v(x)\,dx$.
(d) $V := \mathbb{Q}^d$ and $\langle x, y\rangle := \sum_{k=1}^d x_k y_k$.

Solution:

(a) This is an inner product space, as follows directly from the properties of integrals and complex conjugation. However, it is not complete: the completion is the space $L^2(\mathbb{R}, \mu)$ with respect to the measure $\mu = \frac{dt}{1+t^2}$. To see incompleteness concretely, take $u_\epsilon(t) := e^{-\epsilon t^2} \in V$ for $\epsilon > 0$. We have that $u_\epsilon \to 1$ as $\epsilon \to 0$ with respect to the norm induced by the inner product:

$$\|f\|^2 = \int_{\mathbb{R}} \frac{|f(t)|^2}{1+t^2}\,dt.$$

Indeed

$$\int_{\mathbb{R}} \frac{|u_\epsilon(t) - 1|^2}{1+t^2} dt \to 0$$

by the dominated convergence theorem, since $u_\epsilon \le 1 \in L^2(\mathbb{R}, \mu)$. But then V cannot be complete because $\{u_\epsilon\}$ is Cauchy (being convergent), but $1 \notin V$.

(b) We first check that this is an inner product space, then completeness follows from finite dimensionality. Linearity follows from the linearity of differentiation, so we just have to check that the given bilinear form is positive definite and symmetric. For $p, q \in V$ we write

$$p(X) = \sum_{j=0}^N p_j X^j, \qquad q(X) = \sum_{j=0}^N q_j X^j,$$

and compute

$$\langle p, q\rangle = \sum_{j=0}^N p_j q^{(j)}(0) = \sum_{j=0}^N j!\, p_j q_j,$$

which is clearly symmetric in $p \leftrightarrow q$. We used that $\left(\frac{d}{dx}\right)^j_{x=0} x^k = \delta^{kj}\, k!$. We remark that in the basis $[1, X, \dots, X^N]$ the scalar product is given by the matrix

$$\mathrm{diag}[0!, 1!, 2!, \dots, N!],$$

which is positive definite having positive eigenvalues (recall $0! = 1$). The norm squared is

$$\|p\|^2 = \sum_{j=0}^N j!\, |p_j|^2.$$

(c) No, the given bilinear form is not even well-defined as the integral might diverge. For instance, $u(x) = x^{-1/2}$ satisfies $u \in L^1\big((0,1);\mathbb{R}\big)$, but $\langle u, u\rangle = +\infty$.

(d) It is not even an $\mathbb{R}$-vector space, as the multiplication of a rational number by a real number does not give, in general, a rational number.

If we think $\mathbb{Q}^d$ as a vector space over $\mathbb{Q}$ the scalar product is well defined, so we have a norm, but the resulting space is not complete, for the same reason that $\mathbb{Q}$ is not complete with respect to the absolute value.

Exercise 3.2.
Let

$$V := \big\{u \in C^2((0,1)) \cap C([0,1]) : u', u'' \text{ bounded on } (0,1),\ u(0) = 0\big\}$$

Prove or disprove that the following maps $\|\cdot\|\colon V \to \mathbb{R}$ are norms (no need to check completeness) and determine whether they arise from an inner product.

(a) $\|u\| = \left(\int_0^1 |u''(x)|^2 dx\right)^{1/2}$

(b) $\|u\| = \left(\int_0^1 |u'(x)|^2 dx\right)^{1/2}$

(c) $\|u\| = \left(\int_0^1 |u'(x)|^3 dx\right)^{1/3}$

(d) $\|u\| = \left(\int_0^1 \int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^2} dx dy\right)^{1/2}$

Hint: Recall the Minkowski inequality: for $p \in (1, +\infty)$ and $f, g \in L^p(X, \mu)$, we have $(\int_X |f+g|^p d\mu)^{1/p} \leq (\int_X |f|^p d\mu)^{1/p} + (\int_X |g|^p d\mu)^{1/p}$.

Solution: In all four cases, absolute homogeneity is apparent and the triangle inequality follows by applying the Minkowski inequality to the respective integrals. We check positive definiteness.

(a) $\|\cdot\|$ is not positive definite and hence not a norm. Indeed, the function $f(x) = x$ belongs to V and $\|f\| = 0$, but $f \neq 0$.

(b) If $\|u\| = 0$, then $u' = 0$ almost everywhere, so $u'(x) = 0$ for all $x \in (0,1)$ by the continuity of u'. Thus, u is constant on $(0,1)$ and hence on $[0,1]$ by continuity of u. Since $u(0) = 0$, we must have $u = 0$. This shows that $\|\cdot\|$ defines a norm on V. It is induced by the inner product

$$\langle u,\, v\rangle = \int_0^1 u'(x)\overline{v'(x)}\, dx$$

as can readily be seen.

(c) The exact same argument as in the previous case shows $\|u\| = 0 \implies u = 0$, so $\|\cdot\|$ defines a norm on V. This norm is not induced by an inner product. Indeed, the parallelogram law is not satisfied, as can be seen by taking $u(x) = x$ and $v(x) = \frac{x^2}{2}$. Then

$$\|u+v\|_C^2 + \|u-v\|_C^2 = \left(\int_0^1 |1+x|^3 dx\right)^{2/3} + \left(\int_0^1 |1-x|^3 dx\right)^{2/3} = (15/4)^{2/3} + (1/4)^{2/3}$$
$$2\|u\|_C^2 + 2\|v\|_C^2 = 2\left(\int_0^1 dx\right)^{2/3} + 2\left(\int_0^1 |x|^3 dx\right)^{2/3} = 2 + 2(1/4)^{2/3}$$

and the two values differ.

(d) Note first that $\|\cdot\|$ is well-defined, since $\left|\frac{u(x)-u(y)}{x-y}\right|$ is bounded by $\sup_{t\in(0,1)} |u'(t)|$. If $\|u\| = 0$ then $u(x) - u(y) = 0$ for almost every $(x, y) \in (0,1)^2$, so in fact $u(x) = u(y)$ for all $x, y \in (0,1)$ by continuity, i.e. u is constant. Again $u(0) = 0$ then implies $u = 0$. Thus, $\|\cdot\|$ defines a norm, which is induced by the inner product

$$\langle u,\, v\rangle = \int_0^1 \int_0^1 \frac{(u(x)-u(y))(\overline{v(x)} - \overline{v(y)})}{|x-y|^2}\, dxdy.$$

Exercise 3.3.
Consider the Hilbert space $H := L^2\big((-1,1)\big)$. Apply the Gram-Schmidt algorithm to the ordered set $\{1, x, x^2\} \subset H$, and find three orthonormal polynomials $e_0(x), e_1(x), e_2(x)$.

Solution: The polynomials 1 and x are already orthogonal in H for parity reasons, so normalizing them we find

$$e_0(x) = 1/\sqrt{2}, \quad e_1(x) = \sqrt{\frac{3}{2}}x.$$

We calculate the inner product of x^2 with e_0 and e_1:

$$\langle x^2, e_0(x)\rangle = \frac{1}{\sqrt{2}} \int_{-1}^1 x^2\, dx = \sqrt{2}/3, \quad \langle x^2, e_1(x)\rangle = \sqrt{\frac{3}{2}} \int_{-1}^1 x^3\, dx = 0.$$

Then

$$x^2 - \langle x^2, e_0\rangle e_0 - \langle x^2, e_1\rangle e_1 = x^2 - 1/3$$

is orthogonal to e_0 and e_1. We compute its norm-squared:

$$\int_{-1}^1 (x^2 - 1/3)^2\, dx = 8/45,$$

and normalize to find

$$e_2(x) = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1).$$

Exercise 3.4.
Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Show that a linear subspace of H is itself a Hilbert space with respect to $\langle\cdot,\cdot\rangle$ if and only if it is closed.

Solution: Let $V \subset H$ be a linear subspace. Note that the inner product $\langle\cdot,\cdot\rangle$ on H makes $(V, \langle\cdot,\cdot\rangle)$ into an inner product space. We must show that V is complete with respect to the induced norm if and only if V is closed in H.

Assume that V is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V. Then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy in H (since the norm on V is just the norm on H restricted to V). Thus, by completeness of H, we have $x_n \to x \in H$, i.e. the sequence converges to some $x \in H$. Since $V \subset H$ is closed, it contains all its limit points, so we must have $x \in V$. Thus, $x_n \to x \in V$, so every Cauchy sequence in V converges in V, i.e. V is complete.

For the converse, assume that V is complete. Let $(x_n)_{n\in N}$ be a sequence with $x_n \in V$ for all $n \in \mathbb{N}$ and $x_n \to x \in H$. Since x_n converges in H, it is a Cauchy sequence with respect to the norm. Since $x_n \in V$ and V is complete, x_n must converge to some y in V. Since limits with respect to a norm are unique (follows from positive definiteness of the norm), we must have $y = x \in V$. Thus, V contains all its limit points, i.e. it is closed in H.

Exercise 3.5. ★
This exercise is concerned with a quantitative study of the Cauchy-Schwarz inequality.

(a) Let H be a real inner product space. We write $x \cdot y$ for the inner product of $x, y \in H$ and $|x|$ for the induced norm. Prove the following identity:

$$|x||y| - x\cdot y = \frac{|x||y|}{2}\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^2 \geq 0, \quad \forall\, x, y, \in H.$$

(b) Characterize the set $C \subset H \times H$ of pairs of vectors that saturate the Cauchy-Schwarz inequality, i.e. $x \cdot y = |x||y|$. Plot C in the case $H = \mathbb{R}$.

(c) If x, y are ϵ-close to saturating the Cauchy-Schwarz inequality, that is

$$x \cdot y \geq (1-\epsilon)|x||y|,$$

then how close are x, y to the set C? Find an upper bound for the quantity

$$\operatorname{dist}\big((x,y), C\big)^2 := \inf_{(x',y')\in C} |x - x'|^2 + |y - y'|^2.$$

Solution:

(a) If $x = 0$ or $y = 0$, then both sides of the identity vanish. For $x, y \neq 0$ we expand the norm squared on the right hand side:

$$\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^2 = \Big(\frac{x}{|x|} - \frac{y}{|y|}\Big)\cdot\Big(\frac{x}{|x|} - \frac{y}{|y|}\Big) = 1 - 2\frac{x\cdot y}{|x||y|} + 1 = 2\Big(1 - \frac{x\cdot y}{|x||y|}\Big).$$

Inserting this, the identity follows.

(b) From the above identity, we see that $(x, y) \in C$ if and only if

$$x = 0, \quad \text{or} \quad y = 0 \quad \text{or} \quad \frac{x}{|x|} = \frac{y}{|y|}.$$

For $x, y \neq 0$, we write $x = \alpha\xi, y = \beta\eta$ with $\alpha, \beta > 0$ and $|\xi| = |\eta| = 1$. Then the last equality above implies $\eta = \xi$. The cases $x = 0$ and $y = 0$ can be included by allowing α, β to vanish. Thus, C can be parametrized as follows:

$$C = \big\{(\alpha\xi, \beta\xi) : \alpha \geq 0,\, \beta \geq 0,\, \xi \in H,\, |\xi| = 1\big\}.$$

That is, x, y need to be parallel and oriented the same in order to saturate C.-S. If $H = \mathbb{R}$, then the set $C \subset \mathbb{R}^2$ consists of the (closed) first and third quadrants.

(c) We plug the inequality $x \cdot y \geq (1 - \epsilon)|x||y|$ into the identity from the first part and find

$$\frac{|x||y|}{2}\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^2 = |x||y| - x \cdot y \leq \epsilon|x||y|,$$

which gives

$$\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big| \leq \sqrt{2\epsilon}.$$

Since $x/|x|$ and $y/|y|$ are the directions of the vectors x, y this is telling us that if x, y ϵ-saturate the C.-S. inequality, then the directions of x and y are $O(\sqrt{\epsilon})$ close to each other (on the unit sphere).

This is already a somewhat satisfying answer, but we want to estimate

$$\operatorname{dist}((x, y), C)^2 := \inf_{(x', y') \in C} |x - x'|^2 + |y - y'|^2.$$

In order to do so, we make a judicious guess for $(x', y') \in C$ close to (x, y). Taking $x' = x$, we have that $y' = |y|\frac{x}{|x|}$ satisfies (x', y') with $|y'| = |y|$. We find

$$|x - x'|^2 + |y - y'|^2 = \Big|y - |y|\frac{x}{|x|}\Big|^2 = |y|^2\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|$$

We find a similar expression by choosing $y' = y, x' = |x|\frac{y}{|y|}$. Thus, the infimum satisfies

$$\operatorname{dist}((x, y), C)^2 \leq \min\{|x|^2, |y|^2\}\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^2 \leq 2\epsilon \min\{|x|^2, |y|^2\} \leq 2\epsilon|x||y|,$$

where in the last inequality we used that for any pair of nonnegative numbers $\min\{a, b\} \leq \sqrt{ab}$.

In other words, we have proved the *quantitative* Cauchy-Schwarz inequality:

$$0 \leq \frac{1}{2}\operatorname{dist}((x, y), C)^2 \leq |x||y| - x \cdot y \quad \text{for all } x, y, \in H.$$