Exercise 5.1. ♣
Which of the following statements are true?

(a) If a linear map $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfies $\|Tu\|_{L^2(\mathbb{R})} \leq 100$ for all $u \in L^2(\mathbb{R})$ with $\|u\|_{L^2(\mathbb{R})} \leq \frac{1}{10}$, then T is continuous.

(b) Assume that two bounded linear functionals $\phi, \psi \in L^2([0,1])^*$ agree on $C([0,1])$, i.e. $\phi(u) = \psi(u)$ for all $u \in C([0,1]) \subset L^2([0,1])$. Then $\phi = \psi$.

(c) If ϕ is a continuous linear functional on a Hilbert space H, then $\ker\phi$ is a closed linear subspace of H.

(d) For $x = (x_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ define $(Tx)_k = \log(1/k)x_k$ for all $k \in \mathbb{N}$. Then T defines a bounded linear operator $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$.

(e) For $u \in L^2((0,1))$ define $F(u)(x) = u(x)^2$. Then $F : L^2((0,1)) \to L^2((0,1))$ defines a bounded (non-linear) operator.

Solution:

(a) True. Using absolute homogeneity of the norm, the condition implies that T is bounded, i.e. continuous. More precisely, if $v \in L^2(\mathbb{R})$ then $u := \frac{v}{10\|v\|}$ has norm $\|u\| = \frac{1}{10}$, so by assumption

$$\frac{\|Tv\|}{10\|v\|} = \|Tu\| \leq 100.$$

This gives $\|Tv\| \leq 1000\|v\|$ and since v was arbitrary, this proves that T is bounded.

(b) True. In general, two continuous functions that agree on a dense set must agree everywhere. In this case, $C([0,1])$ is dense in $L^2([0,1])$ (with respect to the L^2 topology) and ϕ, ψ are both continuous (being bounded) with respect to this topology.

More directly: for any $x \in L^2([0,1])$, there exists a sequence $x_n \in C([0,1])$ such that $x_n \to x$ with respect to the $L^2([0,1])$ topology. By continuity of ϕ, ψ we have

$$\phi(x) = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} \psi(x_n) = \psi(x).$$

(c) True. The pre-image of a closed set via a continuous function is closed. In this case $\ker\phi = \phi^{-1}(\{0\})$, ϕ is continuous and $\{0\}$ is closed.

(d) False. Indeed, letting $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$ be the standard basis, we have

$$\|T(e_n)\|_{\ell^2}^2 = \sum_{k=1}^{\infty} |\log(1/k)\delta_{n,k}|^2 = \log^2(n).$$

Thus, $\|T(e_n)\|_{\ell^2} \to \infty$ as $n \to \infty$ while $\|e_n\|_{\ell^2} = 1$ for all $n \in \mathbb{N}$, which contradicts boundedness.

(e) False. F is not well-defined as a map from $L^2((0,1))$ into itself. For instance, the function $u(x) = x^{-\frac{1}{4}}$ lies in $L^2((0,1))$ but $u^2 \notin L^2((0,1))$.

Exercise 5.2.
For a fixed measurable function $a\colon (0,1) \to \mathbb{C}$, consider the multiplication operator

$$M_a : L^2((0,1)) \to L^2((0,1)), \quad M_a u(x) = a(x)u(x).$$

We want to prove that M_a is continuous on $L^2((0,1))$ if and only if $a \in L^\infty(0,1)$, in which case the operator norm satisfies $\|M_a\|_{\text{op}} = \|a\|_{L^\infty(0,1)}$.

(a) Prove the inequality

$$\int_0^1 |a(x)u(x)|^2 \, dx \leq \text{esssup}_{x\in(0,1)}|a(x)|^2 \int_0^1 |u(x)|^2 \, dx,$$

and deduce that $\|M_a\|_{\text{op}} \leq \|a\|_{L^\infty(0,1)}$.

(b) Show that if $E \subset (0,1)$ is measurable with $|E| > 0$, then

$$\frac{\|M_a \mathbf{1}_E\|^2_{L^2(0,1)}}{\|\mathbf{1}_E\|^2_{L^2(0,1)}} = \frac{1}{|E|} \int_E |a(x)|^2 \, dx.$$

(c) By an appropriate choice of the measurable set E in the previous point, prove that $\|M_a\|_{\text{op}} \geq \|a\|_{L^\infty}$.

Hint: Take $E =$"the set where $|a|$ is large" and recall the definition of essential supremum.

Solution:

(a) This is (a very simple case) of Hölder's inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \text{where } p, q \in [1, \infty], \ \frac{1}{p} + \frac{1}{q} = 1,$$

with $f = |a|^2$, $g = |u|^2$, $p = \infty$, $q = 1$. More directly, the essential supremum is characterized by $|a(x)| \leq \text{esssup}|a|$ a.e. So the result follows by integrating w.r.t. x the inequality

$$|a(x)|^2|u(x)|^2 \leq \text{esssup}_{y\in(0,1)}|a(y)|^2 \, |u(x)|^2 \quad \text{for almost every } x \in (0,1).$$

Since $\|M_a u\|^2_{L^2} = \int_0^1 |a|^2|u|^2 \, dx$ and $\|u\|^2_{L^2} = \int_0^1 |u|^2 \, dx$, the inequality is saying

$$\|M_a u\|^2_{L^2} \leq \|a\|^2_{L^\infty}\|u\|^2_{L^2} \quad \text{for all } u \in L^2((0,1)),$$

which means (by definition of the operator norm) that $\|M_a\|_{\text{op}} \leq \|a\|_{L^\infty}$.

(b) This follows from a direct computation:

$$\|M_a \mathbf{1}_E\|^2_{L^2(0,1)} = \int_0^1 |a|^2 \mathbf{1}_E^2 \, dx = \int_E |a|^2 \, dx,$$
$$\|\mathbf{1}_E\|^2_{L^2(0,1)} \, dx = \int_0^1 \mathbf{1}_E^2 = |E|.$$

(c) Pick any $\mu > 0$ such that $|\{|a| \geq \mu\}| > 0$. Then the previous computation with $E := \{|a| \geq \mu\}$ gives

$$\frac{\|M_a \mathbf{1}_E\|^2_{L^2(0,1)}}{\|\mathbf{1}_E\|^2_{L^2(0,1)}} = \frac{1}{|E|}\int_E \underbrace{|a(x)|^2}_{\geq \mu^2}\, dx \geq \mu^2,$$

which proves $\|M_a\|_{\mathrm{op}} \geq \mu$. As this holds for any such μ, we conclude by recalling that the essential supremum of a measurable function is defined as the largest such μ:

$$\|a\|_{L^\infty(0,1)} = \operatorname{esssup}|a| = \sup\big(\{\mu > 0 : |\{|a| \geq \mu\}| > 0\} \cup \{0\}\big).$$

Exercise 5.3.
Prove that each of the following linear operators is bounded on $\ell^2(\mathbb{N})$ (i.e. as an operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$). Illustrate each operator as an infinite matrix with respect to the standard basis vectors $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$.

(a) (Shift operator) $S\colon (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$.

(b) (Diagonal matrix) $M_\lambda\colon (x_1, x_2, x_3, \dots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$, where $\{\lambda_k\}_{k\geq 1}$ is some given sequence such that $\sup_{k\geq 1}|\lambda_k| < \infty$.

(c) $T\colon (x_1, x_2, x_3, \dots) \mapsto (x_1 - x_2, x_2 - x_3, x_3 - x_4, \dots)$.

(d) (Hilbert-Schmidt matrix) For each $k \geq 0$ set $(Ax)_k := \sum_{j\geq 1} A_{k,j} x_j$, where the infinite matrix $\{A_{i,j}\}_{i\geq 1, j\geq 1}$ satisfies

$$\sum_{i,j\geq 1} |A_{i,j}|^2 < \infty.$$

Hint: Apply the Cauchy-Schwarz inequality to the sum $\left|\sum_{j\geq 1} A_{k,j} x_j\right|^2$ for fixed k.

Solution:

(a) The operator is bounded (in fact an isometry) since for any $x \in \ell^2(\mathbb{N})$ we have

$$\|Sx\|^2_{\ell^2(\mathbb{N})} = 0^2 + \sum_{k=1}^\infty |u_k|^2 = \sum_{k=1}^\infty |u_k|^2 = \|u\|^2_{\ell^2(\mathbb{N})}.$$

The infinite matrix of S is

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(b) Denote $C = \sup_{k\geq 1}|\lambda_k|^2 < \infty$. The operator is bounded since for any $x \in \ell^2(\mathbb{N})$ we have

$$\|M_\lambda x\|^2_{\ell^2(\mathbb{N})} = \sum_{k=1}^\infty |\lambda_k x_k|^2 \leq \sum_{k=1}^\infty \big(\sup_{j\geq 1}|\lambda_j|^2\big)|x_k|^2 = \sup_{j\geq 1}|\lambda_j|^2 \sum_{k=1}^\infty |x_k|^2 = C\|u\|^2_{\ell^2(\mathbb{N})}$$

The infinite matrix of M_λ is

$$M_\lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_3 & 0 & \cdots \\ 0 & 0 & 0 & \lambda_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(c) For any $a, b \in \mathbb{C}$, we have

$$|a-b|^2 \leq (|a|+|b|)^2 = |a|^2 + |b|^2 + 2|a||b| \leq 2|a|^2 + 2|b|^2.$$

Thus, for any $x \in \ell^2(\mathbb{N})$ we can estimate

$$\|Tx\|^2_{\ell^2(\mathbb{N})} = \sum_{k=1}^{\infty} |x_{k+1} - x_k|^2 \leq \sum_{k=1}^{\infty} 2|x_{k+1}|^2 + 2|x_k|^2 \leq 4\sum_{k=1}^{\infty} |x_k|^2 = 4\|u\|^2_{\ell^2(\mathbb{N})}.$$

The infinite matrix of T is

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(d) Note that for each $k \geq 1$ and $x \in \ell^2(\mathbb{N})$, we have

$$\sum_{j=1}^{\infty} |A_{kj} x_j| \leq \sum_{j=1}^{\infty} |A_{kj}|^2 + \sum_{j=1}^{\infty} |x_j|^2 \leq \sum_{j,k \geq 1} |A_{kj}|^2 + \sum_{j=1}^{\infty} |x_j|^2,$$

so the sum defining $(Ax)_k$ converges absolutely and Ax is well-defined. We now take any $x \in \ell^2(\mathbb{N})$ and compute

$$\|Ax\|^2_{\ell^2(\mathbb{N})} = \sum_{k=1}^{\infty} |(Ax)_k|^2 = \sum_{k=1}^{\infty} \Big| \sum_{j=1}^{\infty} A_{kj} x_j \Big|^2.$$

For any $N \in \mathbb{N}$, we can apply the Cauchy-Schwarz inequality to find

$$\Big| \sum_{j=1}^{N} A_{kj} x_j \Big|^2 \leq \sum_{j=1}^{N} |A_{kj}|^2 \cdot \sum_{j=1}^{N} |x_j|^2 \leq \sum_{j=1}^{\infty} |A_{kj}|^2 \cdot \sum_{j=1}^{\infty} |x_j|^2.$$

Taking the limit as $N \to \infty$ and summing over k, we have

$$\|Ax\|^2_{\ell^2(\mathbb{N})} = \sum_{k=1}^{\infty} \Big| \sum_{j=1}^{\infty} A_{kj} u_j \Big|^2 \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} |A_{kj}|^2 \cdot \sum_{j=1}^{\infty} |x_j|^2 \right) = \Big(\sum_{kj \geq 1} |A_{kj}|^2 \Big) \|x\|^2_{\ell^2(\mathbb{N})}.$$

The infinite matrix of A is the matrix itself, i.e.

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & \cdots \\ A_{10} & A_{11} & A_{12} & A_{13} & \cdots \\ A_{20} & A_{21} & A_{22} & A_{23} & \cdots \\ A_{30} & A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Exercise 5.4.
Prove the following inequalities and interpret them as the continuity of a suitable linear map between suitable normed vector spaces:

(a) For all $u \in L^2(\mathbb{R})$, we have

$$\int_0^1 |u(t)|^2 \, dt \le \int_{\mathbb{R}} |u(t)|^2 \, dt.$$

(b) For each polynomial $p(X) = p_0 + p_1 X + \ldots + P_k X^N$, we have

$$\max_{x\in[-1,1]} |p(x)| \le \sum_{j=0}^{N} |p_j|.$$

(c) For all $u \in C^1([0,1])$ with $u(0) = 0$, we have

$$\max_{x\in[0,1]} |u(x)| \le \int_0^1 |u'(t)| \, dt.$$

Solution:

(a) The inequality follows from the monotonicity of the integral and $|u|^2 \ge 0$. It can be interpreted as the fact that the restriction operator

$$\rho\colon L^2(\mathbb{R}) \to L^2\big((0,1)\big), \quad u \mapsto u|_{(0,1)},$$

(which is linear) is bounded and, more precisely, $\|\rho\|_{\mathcal{L}(L^2(\mathbb{R});L^2(0,1))} \le 1$.

(b) By the triangle inequality, for each $x \in [-1,1]$ we have

$$|p(x)| \le \sum_{j=0}^{N} |p_j| \underbrace{|x^j|}_{\le 1} \le \sum_{j=0}^{N} |p_j|,$$

so taking the supremum over x we find the desired inequality. We can interpret it as the continuity of the inclusion map

$$\iota\colon (V, \|\cdot\|_1) \to C([-1,1]), \quad p \mapsto p,$$

where V is the (infinite dimensional) vector space of polynomials with norm $\|p\|_1 := \sum_{j=0}^{N} |p_j|$, where $N = N(p)$ is the degree of p, and $C([-1,1])$ is equipped with the $\|\cdot\|_{L^\infty}$ norm as usual.

(c) For all $x \in [0,1]$ and $u \in C^1([0,1])$ with $u(0) = 0$, we have

$$|u(x)| = |u(x) - \underbrace{u(0)}_{=0}| = \left| \int_0^x u'(t) \, dt \right| \le \int_0^x |u'(t)| \, dt \le \int_0^1 |u'(t)| \, dt,$$

so taking the maximum over x we find the desired estimate. If we define

$$X := \{u \in C^1([0,1]) : \text{ with } u(0) = 0\}, \quad \|u\|_X := \|u'\|_{L^1(0,1)},$$

then the inequality is expressing the continuity of the inclusion $X \hookrightarrow C^0([0,1])$. The fact that X is an honest vector space is readily checked, while the fact that $\|\cdot\|_X$ is a norm follows from the linearity of the derivative and

$$\begin{aligned}\|u\|_X = 0 &\Longrightarrow u' = 0 \text{ a.e.} \Longrightarrow u' \equiv 0 \quad (u' \text{ is continuous})\\ &\Longrightarrow u \equiv \text{ const} \Longrightarrow u \equiv 0 \quad (\text{since } u(0) = 0).\end{aligned}$$

Exercise 5.5.
This exercise concerns compactness in infinite dimensional Hilbert spaces.

(a) Show that the closed unit ball $\overline{B_1^{\ell^2(\mathbb{N})}} = \left\{x \in \ell^2(\mathbb{N}) : \|x\|_{\ell^2} \leq 1\right\}$ is not a compact subset of $\ell^2(\mathbb{N})$.

Hint: Find a sequence contained in the closed unit ball with no converging subsequences.

(b) Show that $C = \left\{x = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) : |x_n| \leq \frac{1}{n} \ \forall n \in \mathbb{N}\right\}$ is a compact subset of $\ell^2(\mathbb{N})$. C is known as the Hilbert cube.

Hint: Show that C is totally bounded, i.e. C can be covered by finitely many balls of any fixed radius.

Solution:

(a) Let $e_k = (\delta_{k,n})_{n\in\mathbb{N}}$ be the standard basis of $\ell^2(\mathbb{N})$. Since $\|e_k\|_{\ell^2(\mathbb{N})} = 1$ for all $k \in \mathbb{N}$, the sequence $(e_k)_{k\in\mathbb{N}}$ is contained in the closed unit ball of $\ell^2(\mathbb{N})$. Notice that for all $j, k \in \mathbb{N}$ with $j \neq k$, we have

$$\|e_k - e_j\|_{\ell^2(\mathbb{N})}^2 = \sum_{n=0}^{\infty} |\delta_{k,n} - \delta_{j,n}|^2 = 2.$$

Let e_{k_i} be any subsequence of the basis vectors. Since $\|e_{k_i} - e_{k_j}\|_{\ell^2(\mathbb{N})} = \sqrt{2}$ for all i, j, this is not a Cauchy sequence and hence is non-convergent. But every sequence in a compact subset of a normed vector space must have a converging subsequence.

Notice that the exact same proof shows the non-compactness of the closed unit ball in any Hilbert space with an infinite set of orthonormal vectors.

(b) Note that C is closed. We will show that C is totally bounded, that is for any $\epsilon > 0$, C can be covered by finitely many balls of radius ϵ. Compactness then follows. Thus, fix $\epsilon > 0$. Since the sum $\sum_n \frac{1}{n^2}$ converges, there is some $N \in \mathbb{N}$, such that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \left(\frac{\epsilon}{2}\right)^2.$$

Now consider the set

$$A = \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : |x_n| \leq \tfrac{1}{n} \ \forall 1 \leq n \leq N\right\}.$$

This is a closed and bounded subset of $\mathbb{R}^N$ and thus compact. So we can cover A by finitely many balls of radius $\frac{\epsilon}{2}$:

$$A \subset \bigcup_{i=1}^{m} B\big(y^{(i)}, \tfrac{\epsilon}{2}\big), \quad \text{for some } y^{(i)} \in \mathbb{R}^N.$$

Now for each $1 \leq i \leq m$, let $x^{(i)} = (y_1^{(i)}, \dots, y_N^{(i)}, 0, 0, \dots) \in \ell^2(\mathbb{N})$, i.e. this is just the vector $y^{(i)} \in \mathbb{R}^N$ extended by zeros to an infinite sequence. We claim that

$$C \subset \bigcup_{i=1}^{m} B(x^{(i)}, \epsilon).$$

Indeed, for $x \in C$, we have $(x_1, \dots x_N) \in A$, so there is an i with

$$\|x - y^{(i)}\|_{\mathbb{R}^N}^2 = \sum_{n=1}^{N} \|x_n - y_n^{(i)}\|^2 = \sum_{n=1}^{N} \|x_n - x_n^{(i)}\|^2 < \Big(\frac{\epsilon}{2}\Big)^2.$$

Then

$$\|x - x^{(i)}\|_{\ell^2(\mathbb{N})}^2 = \sum_{n=1}^{\infty} \|x_n - x_n^{(i)}\|^2 = \sum_{n=1}^{N} \|x_n - x_n^{(i)}\|^2 + \sum_{n=N+1}^{\infty} |x_n| < \Big(\frac{\epsilon}{2}\Big)^2 + \sum_{n=N+1} \frac{1}{n^2} < \epsilon^2,$$

by our choice of N.