Exercise 6.1. ♣
For each of the following Hilbert spaces H and maps $\varphi : H \to \mathbb{C}$ determine whether φ defines a continuous linear functional on H.

(a) $H = L^2([-\pi, \pi])$ and $\varphi(f) = c_1(f)$ (the first Fourier coefficient of f).

(b) $H = L^2([-1, 1])$ and $\varphi(f) = f(0)$.

(c) $H = \ell^2(\mathbb{N})$ and $\varphi((x_n)_{n\in\mathbb{N}}) = x_3 + 2x_7$.

(d) $H = L^2([-1, 1])$ and $\varphi(f) = \int_{-1}^{1}(1 + f)^2\, dx$.

(e) $H = L^2(\mathbb{R})$ and $\varphi(f) = \frac{1}{3}\int_{-1}^{1} f\, dx$.

(f) $H = \ell^2(\mathbb{N})$ and $\varphi((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}$.

Solution:

(a) This is a continuous linear functional since by definition it is given by the L^2 pairing $f \to \langle f, e_1\rangle$ with $e_1 = \frac{1}{2\pi}e^{ix}$.

(b) This functional is not even well defined on L^2, as L^2 functions are only defined up to sets of measure zero.

(c) This is a continuous linear functional given by the ℓ^2 pairing with $e_3 + 2e_7 \in \ell^2(\mathbb{N})$, where $\{e_k\}_{k=1}^{\infty}$ is the standard basis of $\ell^2(\mathbb{N})$.

(d) This functional is not linear, since $\varphi(0) \neq 0$.

(e) This is a continuous linear functional given by the L^2 pairing with $\frac{1}{3}\chi_{[-1,1]} \in L^2(\mathbb{R})$.

(f) This is a continuous linear functional given by the ℓ^2 pairing with $(1/n^2)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

Exercise 6.2.
Show that any $f \in C^1([-\pi, \pi])$ with $f(\pi) = f(-\pi)$ and $\int_{-\pi}^{\pi} f(x)\, dx = 0$, satisfies

$$\|f\|_{L^2([-\pi,\pi])} \leq \|f'\|_{L^2([-\pi,\pi])}.$$

Solution: Note that both f and f' are continuous functions on $[-\pi, \pi]$ and hence are contained in $L^2([-\pi, \pi])$. The Fourier coefficients of the derivative satisfy $c_n(f') = inc_n(f)$ for all $n \in \mathbb{N}$. Applying Parseval's identity and using $c_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$, we find

$$\|f\|_{L^2([-\pi,\pi])}^2 = 2\pi \sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n(f)|^2 \leq 2\pi \sum_{n\in\mathbb{Z}} n^2|c_n(f)|^2 = 2\pi \sum_{n\in\mathbb{Z}} |c_n(f')|^2 = \|f'\|_{L^2([-\pi,\pi])}^2.$$

Exercise 6.3.
Let $f \in C(\mathbb{R})$ be a 2π-periodic continuous function satisfying $c_0(f) = 0$.

(a) Show that $F(t) = \int_0^t f(x)\,dx$ is also a 2π-periodic function and determine its Fourier coefficients $c_n(F)$ for all $n \neq 0$.

(b) Determine the Fourier coefficient $c_0(F)$ in terms of the $c_n(f)$.

Solution:

(a) We compute

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(t)\,dt = \int_0^{2\pi} f(t)\,dt = 2\pi c_0(f).$$

Since $c_0(f) = 0$ by hypothesis F is 2π-periodic. We have $F' = f$ by the fundamental theorem of calculus (since f is continuous). Thus, the Fourier coefficients satisfy (integrate by parts)

$$c_n(f) = inc_n(F) \quad \forall\, n \in \mathbb{Z}, \qquad \Longrightarrow \qquad c_n(F) = \frac{c_n(f)}{in} \quad \forall\, n \in \mathbb{Z} \setminus \{0\}.$$

(b) Since $F \in C^1(\mathbb{R})$, its Fourier series converges uniformly. In particular, $F(x) = \sum_{n\in\mathbb{Z}} c_n(F)e^{inx}$ for each $x \in \mathbb{R}$. Applying this to $x = 0$, we find

$$0 = F(0) = \sum_{n\in\mathbb{Z}} c_n(F) = c_0(F) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{c_n(f)}{in}.$$

Thus,

$$c_0(F) = -\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{c_n(f)}{in}.$$

Exercise 6.4.

(a) Is there an element of $L^2((0, 2\pi))$ whose Fourier series is

$$\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)}\,?$$

(b) Show that the sequence above converges pointwise for each $x \in (0, 2\pi)$.

Hint: Use Dirichlet's test.

Solution:

(a) Note that

$$\sum_{n=2} \frac{\sin(nx)}{\log(n)} = \sum_{n=2} \frac{1}{2i\log(n)}(e^{inx} - e^{-inx}).$$

Thus, the Fourier coefficients of such a function would be $c_0 = c_1 = c_{-1} = 0$ and

$$c_n(f) = \frac{\mathrm{sgn}(n)}{2i\log(|n|)}, \quad |n| > 2.$$

If these were the Fourier coefficients of an L^2 function, then by Parseval's identity the sequence

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)^2}$$

would have to converge. Since this sum is divergent, there is no such function.

(b) We show that for any fixed $x \in (0, 2\pi)$ the partial sums $S_N = \sum_{n=2}^{N} \sin(nx)$ remain bounded independently of N. Since $\log(n)^{-1}$ is a monotone decreasing sequence converging to 0, Dirichlet's test then implies that $\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)}$ converges. Note that

$$\sum_{n=1}^{N} \sin(nx) = \frac{1}{2i} \sum_{n=0}^{N} \Big(e^{inx} - e^{-inx} \Big).$$

Summing a geometric series we find

$$\sum_{n=0}^{N} e^{inx} = \frac{1 - e^{i(N+1)x}}{1 - e^{ix}}, \quad \Longrightarrow \quad \Big| \sum_{n=0}^{N} e^{inx} \Big| \leq \frac{2}{|1 - e^{ix}|},$$

so we can estimate

$$\Big| \sum_{n=0}^{N} \sin(nx) \Big| \leq \frac{1}{|1 - e^{ix}|} + \frac{1}{|1 - e^{ix}|}.$$

Thus, the partial sums S_N are bounded independently of N (recall that $x \notin 2\pi\mathbb{Z}$).

Remark: In fact, using Dirichlet's test one can show that on any compact set in $(0, 2\pi)$ the sequence $\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log(n)}$ converges uniformly to a continuous function. Nevertheless, it is not the Fourier series of an L^2 function.

Exercise 6.5.

The purpose of this exercise is to prove the Riemann–Lebesgue lemma in the special case of a characteristic function of a bounded interval in $\mathbb{R}$.

Let $p : \mathbb{R} \to \mathbb{C}$ be a bounded T-periodic function. Let $f = \mathbf{1}_{[a,b]}$ be the characteristic function of a bounded interval $[a, b] \subset \mathbb{R}$ for some $a < b$. Show that

$$\lim_{x \to \pm\infty} \int_{\mathbb{R}} f(t) p(xt)\, dt = \mu \int_{\mathbb{R}} f(t)\, dt, \tag{1}$$

where μ is the average of p over one period

$$\mu = \frac{1}{T} \int_0^T p(t)\, dt.$$

Remark: The Riemann–Lebesgue lemma says that (1) in fact holds for any $f \in L^1(\mathbb{R})$. How might one use this exercise to prove the more general version?

Solution: Without loss of generality we can assume that $\mu = 0$. The case $\mu \neq 0$ then follows by replacing p with $p - \mu$. Note that

$$\int_{\mathbb{R}} f(t)p(xt)\,dt = \int_{\mathbb{R}} \mathbf{1}_{[a,b]} p(xt)\,dt = \int_a^b p(xt)\,dt.$$

We first consider the case $x \to +\infty$. Thus, let $x > 0$ and make the substitution $u = xt$. Then

$$\int_a^b p(xt)\,dt = \frac{1}{x}\int_{ax}^{bx} p(u)\,du.$$

By the periodicity of p, the integral of p over any interval of length T is $T\mu = 0$, i.e. we have

$$\int_c^{c+T} p(u)\,du = \int_0^T p(u)\,du = 0, \quad \text{for any } c \in \mathbb{R}.$$

For fixed $x > 0$, let $m \in \mathbb{N}$ be the largest integer such that $mT \leq x(b-a)$. We can partition the interval $[xa, xb]$ as

$$[xa, xb] = \bigcup_{k=1}^{m} \big[xa + (k-1)T, xa + kT\big) \cup \big[xa + mT, xb\big],$$

where $\big|[xa + mT, xb]\big| \leq T$. We then have

$$\int_{ax}^{bx} p(u)\,du = \sum_{k=1}^{m} \int_{xa+(k-1)T}^{xa+kT} p(u)\,du + \int_{xa+mT}^{xb} p(u)\,du = \int_{xa+mT}^{xb} p(u)\,du.$$

We estimate

$$\Big|\int_{xa+mT}^{xb} p(u)\,du\Big| \leq \int_{xa+mT}^{xb} |p(u)|\,du \leq \int_{xa+mT}^{xa+(m+1)T} |p(u)|\,du = \int_0^T |p(u)|\,du \leq T \sup_{t\in[0,T]} |p(t)| < \infty$$

by the boundedness of p. It follows that for all $x > 0$ we have

$$\Big|\int_{\mathbb{R}} f(t)p(xt)\,dt\Big| = \Big|\int_a^b p(xt)\,dt\Big| \leq \frac{1}{|x|}\int_0^T |p(u)|\,du \xrightarrow{x\to\infty} 0.$$

For $x < 0$ the proof is the same, except that one considers the interval $[xb, xa]$.

Remark: It is evident that the result proved here for f a characteristic function of an interval also applies when f is a step function (finite linear combinations of characteristic functions). By approximating a function $f \in L^1(\mathbb{R})$ by step functions, one can show that with p as in the exercise, we in fact have

$$\lim_{x\to\pm\infty} \int_{\mathbb{R}} f(t)p(xt)\,dt = \mu \int_{\mathbb{R}} f(t)\,dt \quad \text{for all } f \in L^1(\mathbb{R}).$$

This result is known as the Riemann–Lebesgue lemma.

Exercise 6.6.
Use the Riemann-Lebesgue lemma (see the Remark in Exercise 6.5 or Lemma 2.32 in the lecture notes) to compute the following limits.

(a) Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure $|A| < \infty$. Compute

$$\lim_{m\to\infty} \int_A \sin^2(mx)\, dx.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic of period 1. Compute

$$\lim_{\epsilon\to 0^+} \sqrt{\epsilon} \int_{\mathbb{R}} f(x) e^{-\epsilon\pi x^2}\, dx.$$

Solution:

(a) Let $f = \mathbf{1}_A$ be the characteristic function of the set A. Since A has finite Lebesgue measure, we have $f \in L^1(\mathbb{R})$. Let $p(x) = \sin^2(x)$. Then p is 2π-periodic and the average of p over one period is

$$\mu = \frac{1}{2\pi} \int_0^{2\pi} \sin^2(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \cos(2x)}{2}\, dx = \frac{1}{2},$$

where we used that $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$ and $\int_0^{2\pi} \cos(2x)\, dx = 0$. Thus, the Riemann-Lebesgue lemma gives

$$\lim_{m\to\infty} \int_A \sin^2(mx)\, dx = \lim_{m\to\infty} \int_{\mathbb{R}} f(x) p(mx)\, dx = \mu \int_{\mathbb{R}} f(x)\, dx = \frac{1}{2} \int_{\mathbb{R}} \mathbf{1}_A\, dx = \frac{|A|}{2}.$$

(b) We make the substitution $t = \sqrt{\epsilon} x$ and find

$$\sqrt{\epsilon} \int_{\mathbb{R}} f(x) e^{-\epsilon\pi x^2}\, dx = \int_{\mathbb{R}} f\big(\tfrac{t}{\sqrt{\epsilon}}\big) e^{-\pi t^2}\, dt.$$

Since $e^{-\pi t^2} \in L^1(\mathbb{R})$ and f is 1-periodic, we can apply the Riemann-Lebesgue lemma to find

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}} f\big(\tfrac{t}{\sqrt{\epsilon}}\big) e^{-\pi t^2}\, dt = \int_{\mathbb{R}} e^{-\pi x^2}\, dx \int_0^1 f(x)\, dx = \int_0^1 f(x)\, dx,$$

where we used that the Gaussian integral satisfies $\int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$.