Exercise 7.1. ♣
Which of the following series is the Fourier series of a function $f \in L^2((-T,T))$ for an appropriate choice of $T > 0$?

(a) $\sum_{n=1}^{\infty} \frac{\cos(nx)}{\sqrt{n}}$.

(b) $\sum_{n=2}^{\infty} \frac{\sin(n\pi x)}{\sqrt{n}\log(n)}$.

(c) $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} + \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$.

(d) $\sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n\sqrt{n}}$.

Solution: One must check that the series converge in $L^2((-T,T))$, where $2T$ is the period of the respective trigonometric polynomials. (Note that the trigonometric polynomials are of course also periodic of period $k \cdot 2T$ for any $k \in \mathbb{N}$, but this perspective does not change anything, since for these series with $2T$-periodic terms convergence in $L^2((-T,T))$ and $L^2((-kT,kT))$ are equivalent.) Indeed, if the series does not converge in L^2, then it cannot be the Fourier series of an L^2 function. Conversely, if the series does converge in L^2 then the limit is an L^2-function whose Fourier series, by uniqueness, is given by the expression in question. Since $\left\{\frac{1}{\sqrt{2T}} e^{\frac{\pi}{T} nx}\right\}_{n\in\mathbb{Z}}$, respectively $\left\{\frac{1}{\sqrt{2T}}, \frac{1}{\sqrt{T}}\cos(\frac{\pi}{T}nx), \frac{1}{\sqrt{T}}\sin(\frac{\pi}{T}nx)\right\}_{n\in\mathbb{N}}$, are orthonormal bases of $L^2((-T,T))$, convergence in L^2 is equivalent to the coefficients being square-summable.

(a) This is not the Fourier series of an L^2 function. Indeed, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(b) This is the Fourier series of a function in $L^2((-1,1))$. Indeed, the series $\sum_{n=2}^{\infty} \frac{1}{n\log(n)^2}$ is convergent. This can be seen from convergence of the integral $\int_2^{\infty} \frac{dx}{x\log(x)^2} = \int_{\log(2)}^{\infty} \frac{dy}{y^2} < \infty$.

(c) This is the Fourier series of a function in $L^2((-\pi,\pi))$. Indeed, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are both convergent.

(d) This is the Fourier series of a function in $L^2((-\frac{1}{2},\frac{1}{2}))$. Indeed, the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent. In fact, this is even the Fourier series of a continuous function: the convergence of $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ implies convergence of the Fourier series with respect to the uniform norm, hence in the space of continuous functions.

Exercise 7.2.
Let $f, g \in C(\mathbb{R})$ be 2π-periodic continuous functions. Compute the Fourier coefficients of each of the following 2π-periodic functions in terms of the Fourier coefficients of f, g.

(a) $f_\tau(x) := f(x-\tau)$ for some $\tau \in \mathbb{R}$.

(b) $f \cdot g(x) := f(x)g(x)$.

(c) $f * g(x) := \int_{-\pi}^{\pi} f(x-t)g(t)\,dt$.

Solution:

(a) We compute

$$\begin{aligned} c_n(f_\tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\tau(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\tau)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi-\tau}^{\pi-\tau} f(y)e^{-in(y+\tau)}\,dy = \frac{e^{-in\tau}}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy = e^{-in\tau}c_n(f), \end{aligned}$$

where we made the change of variable $y = x - \tau$ and used the fact that both f and $e^{in(\cdot)}$ are 2π-periodic.

(b) Writing $g(x) = \sum_{m\in\mathbb{Z}} c_m(g)e^{imx}$ as a Fourier series, we find

$$\begin{aligned} c_n(f\cdot g) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m\in\mathbb{Z}} f(x)c_m(g)e^{imx}e^{-inx}\,dx \\ &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} c_m(g)\int_{-\pi}^{\pi} f(x)e^{-i(n-m)x}\,dx = \sum_{m\in\mathbb{Z}} c_{n-m}(f)c_m(g). \end{aligned}$$

Notice that exchanging the order of integration and the sum is justified since $\sum_{m\in\mathbb{Z}} c_m(g)e^{imx}$ converges in $L^2((-\pi,\pi))$ and integration against the L^2 function $f(x)e^{-inx}$ defines a continuous linear functional on $L^2((-\pi,\pi))$. Equivalently, one could also use Fubini's theorem, since applying Cauchy-Schwarz to the integral, we find

$$\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi}|f(x)c_m(g)|\,dx \le \sum_{m\in\mathbb{Z}} 2\pi|c_m(g)|^2\int_{-\pi}^{\pi}|f(x)|^2\,dx < \infty.$$

(c) We compute

$$\begin{aligned} c_n(f*g) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-t)g(t)e^{-inx}\,dtdx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-t)e^{-in(x-t)}g(t)e^{-int}\,dxdt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi-t}^{\pi-t} f(y)e^{-iny}g(t)e^{-int}\,dydt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy\int_{-\pi}^{\pi} g(t)e^{-int}\,dt \\ &= 2\pi c_n(f)c_n(g), \end{aligned}$$

where we used Fubini's theorem to exchange the order of integration, applied a change of variable $y = x - t$ and used the periodicity of f.

Exercise 7.3.

The goal of this exercise is to show that every function in $L^2((0,\pi);\mathbb{R})$ can be expressed as a real Fourier series of sines.

(a) Show that if $f \in L^2((-\pi,\pi);\mathbb{R})$ is odd then its Fourier coefficients $c_n(f)$ are purely imaginary and $c_0(f) = 0$;

(b) Show that if $f \in L^2((-\pi,\pi);\mathbb{R})$ is odd then its N-th Fourier partial sum satisfies

$$S_N f(x) = \sum_{n=1}^{N} 2ic_n(f)\sin(nx).$$

(c) Given $g \in L^2((0,\pi);\mathbb{R})$ show that $\tilde{S}_N g \to g$ in $L^2((0,\pi);\mathbb{R})$ as $N \to \infty$, where

$$\tilde{S}_N g(x) := \sum_{n=1}^{N} a_n(g)\sin(nx), \qquad a_n(g) := \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx.$$

(d) Conclude that $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$ is a Hilbert basis for $L^2((0,\pi);\mathbb{R})$.

Solution:

(a) We compute

$$\begin{aligned}
c_n(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_0^\pi f(x)e^{-inx}\,dx + \frac{1}{2\pi}\int_{-\pi}^0 f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_0^\pi f(x)e^{-inx}\,dx + \frac{1}{2\pi}\int_0^\pi f(-x)e^{inx}\,dx \\
&= \frac{1}{2\pi}\int_0^\pi f(x)(e^{-inx} - e^{inx})\,dx \\
&= \frac{-i}{\pi}\int_0^\pi f(x)\sin(nx)\,dx,
\end{aligned}$$

where we used the change of variables $x \mapsto -x$. Since $f(x)\sin(x) \in \mathbb{R}$ by assumption on f, we conclude that $c_n(f) \in i\mathbb{R}$. Furthermore, we see that $c_0(f) = 0$.

(b) Note that for $n \in \mathbb{Z}$ we have

$$c_{-n}(f) = \frac{-i}{\pi}\int_0^\pi f(x)\sin(-nx)\,dx = \frac{i}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = -c_n(f).$$

Thus, we find

$$\begin{aligned}
S_N(f) &= \sum_{n=-N}^{N} c_n(f)e^{inx} = c_0(f) + \sum_{n=1}^{N}\Big(c_n(f)e^{inx} + c_{-n}(f)e^{-inx}\Big) \\
&= \sum_{n=1}^{N} c_n(f)\Big(e^{inx} - e^{-inx}\Big) = \sum_{n=1}^{N} 2ic_n(f)\sin(nx)
\end{aligned}$$

(c) We extend g to an odd function on $(-\pi,\pi)$. Let $f : (-\pi,\pi) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} g(x), & \text{if } x \in (0,\pi) \\ -g(-x), & \text{if } x \in (-\pi,0). \end{cases}$$

(Note that the value at $x = 0$ is irrelevant since L^2 functions are only defined up to null sets anyway.) Then $f \in L^2((-\pi,\pi);\mathbb{R})$ is odd and we may use the previous subquestion to compute the N-th partial sum of its Fourier series:

$$S_N f(x) = \sum_{n=1}^{N} 2ic_n(f)\sin(nx) = \sum_{n=1}^{N} a_n(g)\sin(nx) \quad \text{for a.e. } x \in (-\pi,\pi),$$

where we used the first subquestion to compute

$$2ic_n(f) = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx = a_n(g).$$

Thus, we have $S_N f|_{(0,\pi)} = \tilde{S}_N g$. Since $S_N f \to f$ in $L^2((-\pi,\pi))$ (Corollary 2.7 in the lecture notes), we conclude convergence:

$$\left\| \tilde{S}_N g - g \right\|_{L^2(0,\pi)} = \left\| S_N f|_{(0,\pi)} - f|_{(0,\pi)} \right\|_{L^2(0,\pi)} \le \|S_N f - f\|_{L^2(-\pi,\pi)} \xrightarrow{N\to 0} 0.$$

(d) The previous subquestion shows that $\operatorname{Span}\left\{\sqrt{2/\pi}\sin(kx)\right\}_{n\in\mathbb{N}}$ is dense in $L^2((0,\pi);\mathbb{R})$. It remains to show L^2-orthonormality. Recall the identity

$$\sin(x)\sin(y) = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big).$$

For $n, m \ge 1$ with $n \ne m$, we compute

$$\begin{aligned}
\left\langle \sqrt{2/\pi}\sin(nx), \sqrt{2/\pi}\sin(mx) \right\rangle_{L^2((-\pi,\pi))} &= \frac{2}{\pi}\int_0^\pi \sin(nx)\sin(mx)\,dx \\
&= \frac{1}{\pi}\int_0^\pi \big(\cos((n-m)x) - \cos((n+m)x)\big)\,dx \\
&= \frac{1}{\pi}\left(\left[\frac{1}{n-m}\sin((n-m)x)\right]_0^\pi + \left[\frac{1}{n+m}\sin((n+m)x)\right]_0^\pi\right) \\
&= 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\left\| \sqrt{2/\pi}\sin(nx) \right\|^2_{L^2((-\pi,\pi))} &= \frac{2}{\pi}\int_0^\pi \sin(nx)^2\,dx \\
&= \frac{1}{\pi}\int_0^\pi \big(1 - \cos(2nx)\big)\,dx \\
&= \frac{1}{\pi}\left(\pi - \left[\frac{1}{2n}\sin(2nx)\right]_0^\pi\right) \\
&= 1.
\end{aligned}$$

Exercise 7.4.

Note that the Fourier coefficients $c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$ are in fact well-defined for any $f \in L^1((-\pi,\pi))$. However, the Fourier series of an L^1 function does not necessarily converge. The goal of this exercise is to show that if $c_n(f) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.

(a) For $f \in L^1((-\pi,\pi))$, we define

$$f_r(x) = \sum_{n\in\mathbb{Z}} c_n(f) r^{|n|} e^{inx}, \quad \text{for } 0 \le r < 1.$$

Show that f_r is well-defined for all $r \in [0,1)$ and can be written as a convolution:

$$f_r(x) = P_r * f(x) := \int_{-\pi}^{\pi} P_r(x-y) f(y)\,dy, \quad \text{where} \quad P_r(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|} e^{inx}.$$

Remark: P_r is known as the Poisson kernel.

(b) Show that

$$P_r(x) = \frac{1}{2\pi} \cdot \frac{1-r^2}{1-2r\cos(x)+r^2}.$$

(c) Show that the family $(P_r)_{0\le r<1}$ is an approximate identity. That is,

- $P_r \ge 0$ and $\int_{-\pi}^{\pi} P_r(x)\,dx = 1$ for all $r \in [0,1)$.
- for all $\delta > 0$ we have $\int_{\{\delta<|x|<\pi\}} P_r(x)\,dx \to 0$ as $r \to 1^-$.

(d) Show that f_r converges to f in $L^1((-\pi,\pi))$ as $r \to 1^-$.
Hint: Use exercise 13.7 from Analysis III.

(e) Conclude that if f satisfies $c_n(f) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ a.e.

Solution:

(a) Note that

$$|c_n(f)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\right| \le \frac{1}{2\pi}\|f\|_{L^1((-\pi,\pi))}, \quad \forall\, n \in \mathbb{Z}.$$

We thus have $\sum_{n\in\mathbb{Z}} |c_n(f) r^{|n|} e^{inx}| \le \sum_{n\in\mathbb{Z}} r^{|n|} < \infty$ for all $r \in [0,1)$ by convergence of the geometric series. Thus, the sum defining $f_r(x)$ converges for each $x \in (-\pi,\pi)$. Using the definition of Fourier coefficients, we find

$$f_r(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} f(y)e^{-iny} r^{|n|} e^{inx}\,dy = \int_{-\pi}^{\pi} f(y) \cdot \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|} e^{in(x-y)}\,dy = P_r * f(x),$$

where we used Fubini to exchange the sum and the integral.

(b) Fix r and x and note that

$$\sum_{n\in\mathbb{Z}} r^{|n|} e^{inx} = \sum_{n=0}^{\infty} r^n e^{inx} + \sum_{n=1}^{\infty} r^n e^{-inx} = \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega}^n,$$

where we defined the complex number $\omega = re^{ix}$. Since $|\omega| < 1$, the geometric series above are convergent and we find

$$\sum_{n=0}^{\infty} \omega^n + \sum_{n=1} \overline{\omega}^n = \frac{1}{1-\omega} + \Big(\frac{1}{1-\overline{\omega}} - 1\Big) = \frac{1}{1-\omega} + \frac{\overline{\omega}}{1-\overline{\omega}} = \frac{1-|\omega|^2}{1-2\mathrm{Re}(\omega)+|\omega|^2}.$$

We conclude by inserting the definition of ω.

(c) Since $|r\cos(x)| \leq r$ and $0 \leq r < 1$, the non-negativity of P_r can be seen from the explicit formula. Using the fact that $\int_{-\pi}^{\pi} e^{inx}$ for all $n \neq 1$, we find from the definition of P_r:

$$\int_{-\pi}^{\pi} P_r(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} r^{|n|} e^{inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} r^0\, dx = 1,$$

where we used Fubini to exchange the sum and the integral. For the second property, we write

$$P_r(x) = \frac{1-r^2}{(1-r)^2 + 2r(1-\cos(x))} \leq \frac{1-r^2}{2r(1-\cos(x))}.$$

Fix $\delta > 0$ and note that $1-\cos(x) \geq c$ for some c and all $\delta < |x| < \pi$. Taking $r \geq \frac{1}{2}$, we find $P_r(x) \leq \frac{1}{c}(1-r^2)$ for all $x \in (-\pi,\pi)$. Thus, $P_r(x) \to 0$ as $r \to 1$ uniformly on $\{\delta < |x| < \pi\}$ and the exercise follows.

(d) We use the result of exercise 13.7 (d) from Analysis III on approximate identities. Since this result is formulated in terms of convolution on $\mathbb{R}$, we rewrite f_r as such. Extending f periodically to $(-2\pi, 2\pi)$ and setting $f = 0$ on $\mathbb{R} \setminus (-2\pi, 2\pi)$, we have for $x \in (-\pi,\pi)$

$$\begin{aligned} P_r * f(x) &= \int_{-\pi}^{\pi} P_r(x-y) f(y)\, dy = \int_{x-\pi}^{x+\pi} P_r(y) f(x-y)\, dy = \int_{x-\pi}^{x+\pi} P_r(y) f(x-y)\, dy \\ &= \int_{-\pi}^{\pi} P_r(y) f(x-y)\, dy = \int_{\mathbb{R}} P_r(y) f(x-y)\, dy = f * P_r(x), \end{aligned}$$

where we used the change of variables $y \to x-y$ and the periodicity of f and P_r. In the final equality, we set $P_r(y) = 0$ outside $(-\pi,\pi)$. Note that f extended in this way lies in $L^1(\mathbb{R})$ and, by the previous subquestion $(P_r)_{0\leq r<1}$ is an approximate identity in the sense exercise 13.7. Applying 13.7 (d), we see that $f_r = P_r * f \to f$ in $L^1(\mathbb{R})$ as $r \to 1$, and thus also in $L^1((-\pi,\pi))$. Note that $(1-r)$ corresponds to ε in exercise 13.7.

Remark: The convergence $f_r \to f$ in L^1 also implies that there is a subsequence $(r_k)_{k\in\mathbb{N}}$ with $r_k \to 1$ such that $f_{r_k}(x) \to f(x)$ for almost every $x \in (-\pi,\pi)$. In fact, one can show the stronger result $f_r(x) \to f(x)$ for almost every $x \in (-\pi,\pi)$ (without needing to pick a subsequence), see Theorem 2.1 in Chapter 3 of "Real Analysis: Measure Theory, Integration, and Hilbert Spaces" by Stein and Shakarchi.

(e) If $f \in L^1((-\pi,\pi))$ satisfies $c_n(f) = 0$ for all n, then $f_r(x) = 0$ for each $r \in [0,1)$. Since $f = \lim_{r\to 1} f_r$ in L^1, we have $f = 0$ in $L^1((-\pi,\pi))$, i.e. $f(x) = 0$ almost everywhere.

Exercise 7.5.
Let $f \in L^1((-\pi,\pi))$ and let $c_n(f)$ be its Fourier coefficients.

(a) Show that if $\sum_{n\in\mathbb{Z}} |c_n(f)|^2 < \infty$, then in fact $f \in L^2((-\pi,\pi))$.
Hint: Show that the sequence of Fourier partial sums S_N is Cauchy in L^2 and use the previous exercise.

(b) Show that if $\sum_{n\in\mathbb{Z}} |c_n(f)| < \infty$, then in fact[1] $f \in C_{per}([-\pi,\pi])$.
Hint: Show that the sequence of Fourier partial sums S_N is Cauchy in the uniform norm and use the previous exercise.

Solution:

(a) We write $S_N f(x) = \sum_{|n|\le N} c_n(f)e^{inx}$ for the N-th Fourier partial sum. Note that $S_N f$ is trivially in $L^2((-\pi,\pi))$, since it is a finite sum of L^2 functions. Let $M, N \in \mathbb{N}$ with $M > N$. Then by the orthonormality of $\{e^{in(\cdot)}\}_{n\in\mathbb{Z}}$, we have

$$\|S_M - S_N\|^2_{L^2((-\pi,\pi))} = \Big\| \sum_{N<|n|\le M} c_n(f)e^{in(\cdot)} \Big\|^2_{L^2((-\pi,\pi))} = \sum_{N<|n|\le M} |c_n(f)|^2 \longrightarrow 0$$

as $N, M \to \infty$, since by assumption this is the tail of a convergent series. By completeness of $L^2((-\pi,\pi))$, the sequence $(S_N)_{N\in\mathbb{N}}$ has a limit $\tilde{f}$ in $L^2((-\pi,\pi))$. Moreover, for every $n \in \mathbb{N}$, we have

$$c_n(\tilde{f}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} S_N f(x)e^{-inx}\,dx = \frac{1}{2\pi}\lim_{N\to\infty}\int_{-\pi}^{\pi} S_N f(x)e^{-inx}\,dx = c_n(f), \tag{1}$$

where we used the L^2 continuity of integration against $e^{-in(\cdot)}$. Thus, $f - \tilde{f} \in L^1((-\pi,\pi))$ satisfies $c_n(f - \tilde{f}) = 0$ for every $n \in \mathbb{Z}$. By the previous exercise, we must have $f = \tilde{f}$ almost everywhere.

(b) The solution is similar to the previous point, with the only difference that now we check that $(S_N)_{N\in\mathbb{N}}$ is Cauchy with respect to the uniform norm. As a finite sum of continuous functions $S_N \in C_{per}([-\pi,\pi];\mathbb{C})$ for every $N \in \mathbb{N}$. For $M > N$, we use $|e^{inx}| = 1$ to find

$$\|S_M - S_N\|_\infty = \sup_{x\in[-\pi,\pi]} \Big| \sum_{N<|n|\le M} c_n(f)e^{inx} \Big| \le \sum_{N\le|n|\le M} |c_n(f)| \longrightarrow 0$$

as $N, M \to \infty$, since again by assumption this is the tail of a convergent series. $C_{per}([-\pi,\pi];\mathbb{C})$ is complete with respect to the supremum norm and S_N is Cauchy, so $S_N \to \tilde{f} \in C_{per}([-\pi,\pi];\mathbb{C})$. (Note that the subspace $C_{per}([-\pi,\pi]) \subset C([-\pi,\pi])$ defined by $f(\pi) = f(-\pi)$ is closed and hence complete.) As before, it remains to show that $f = \tilde{f}$ almost everywhere, which follows from $c_n(\tilde{f}) = c_n(f)$ and the previous exercise. Note that here we use $\left|\int_{-\pi}^{\pi} \varphi(x)e^{-inx}\,dx\right| \le 2\pi\|\varphi(x)\|_\infty$ to justify exchanging the limit and the integral in (1).

[1] This is a slight abuse of notation. More precisely: there exists a (necessarily unique) continuous and periodic $\tilde{f}$ such that $\tilde{f} = f$ a.e.