Exercise 8.1. ♣
For each of the following functions defined on $[-\pi, \pi]$,

- $f_1(x) = \tan(\sin(x))$
- $f_2(x) = |x|^{2/3}$
- $f_3(x) = x$
- $f_4(x) = e^{-x^2}$
- $f_5(x) = |x|^{-1/2}$

answer the following questions using the theorems seen in class. If none of the convergence theorems applies, that's still a valid answer.

(a) Are the Fourier coefficients well-defined?

(b) Is it true that $S_N(f) \to f$ in L^2?

(c) Is it true that $S_N(f)(x) \to f(x)$ for all $x \in [-\pi, \pi]$? **Hint**: Recall Theorem 2.28.

(d) Is it true that $S_N(f) \to f$ in C_{per}? **Hint**: Recall Corollary 2.20.

Solution:

(a) All functions f_k, $k = 1, \ldots, 5$ are in $L^1(-\pi, \pi)$, so the Fourier coefficients are well-defined.

(b) L^2 convergence holds for all functions that are of class L^2, thus it is valid for f_1, f_2, f_3, f_4. It is not valid for f_5, since $f_5 \notin L^2$ (if the convergence was true, it would imply that $f_5 \in L^2$).

(c) For f_1, f_4 the convergence is uniform (thus pointwise) since they are continuous with continuous derivatives (including extrema!). For f_2 pointwise convergence still holds, since f is Hölder continuous in $[-\pi, \pi]$. As per f_3, the function is clearly C^1 in the interior of $(-\pi, \pi)$ so we have pointwise convergence there. We cannot possibly have pointwise convergence at $x = \pm\pi$, simply because $f_3(\pi) \neq f_3(-\pi)$, but $S_N(f_3)(\pi) = S_N(f_3)(-\pi)$ for all N since $\{S_N(f)\}_N$ are 2π periodic functions. Finally, for f_5 we have pointwise convergence only outside of 0, since the limit is not defined in 0 and the function is locally Lipschitz outside of the origin.

(d) For f_1, f_4 we already observed that the convergence is uniform. Since all these functions are continuous on periodic, we can say that the convergence happens in C_{per}. The function f_2 is not piecewise C^1 (there is no partition of $[-\pi, \pi]$ into closed intervals such that f_2 is C^1 on each closed interval) thus we cannot apply any of the results we have seen that ensure uniform convergence. On the other hand there is no obvious contradiction in the fact that $S_N(f_2) \to f_2$ uniformly. Thus in this case we cannot apply our results directly, and this is a correct answer for the sake of the exercise. Finally, since f_3 and f_5 are not continuous and periodic, they cannot be approximated uniformly with their partial Fourier sums (the partial Fourier sums lie in C_{per} and uniform limit of C_{per} functions lies in C_{per}).

Exercise 8.2.

(a) Construct $f\colon [-\pi,\pi] \to \mathbb{R}$ which is continuous, but not Hölder at $x=0$.
Hint: try using $1/\log(x/2\pi)$.

(b) Let V be the vector space of sequences $f\colon \mathbb{N} \to \mathbb{R}$ such that

$$\|f\|_V := \big(\sum_{k\geq 1} k^2 |f(k)|^2\big)^{1/2} < \infty.$$

Can you choose a scalar product on V that makes V a Hilbert space?
Hint: try to construct an L^2 space over $\mathbb{N}$ with the right measure.

(c) Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that

$$\|f\|_V := \sum_{k\geq 1} k|f(k)| < \infty.$$

Can you choose a scalar product on V that makes V a Hilbert space?

(d) Explain the difference between the following spaces of (real) functions and provide elements that fit in one but none of the others:

$$C_{per}([-\pi,\pi];\mathbb{R}), \quad C^2_{per}([-\pi,\pi];\mathbb{R}), \quad C((-\pi,\pi);\mathbb{R}), \quad C([-\pi,\pi];\mathbb{R}).$$

Solution:

(a) Let $f(t) = 1/\log(\frac{t}{2\pi})$ for $t \in (0,\pi]$ and $f(t)=0$ for $t \in [-\pi, 0]$. Then, f is continuous in $[-\pi,\pi]$. Suppose by contradiction that it is also Hölder continuous in $[-\pi,\pi]$; this means that there exists $\alpha \in (0,1)$, $C>0$ such that

$$\sup_{x\neq y \in [-\pi,\pi]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq C.$$

Setting $y=0$ we would obtain for any $x \in (0,\pi]$ that

$$0 < \frac{1}{C} \leq \frac{x^\alpha}{|f(x)|} = |\log(x/2\pi)| x^\alpha,$$

which is in contradiction to the fact that $\lim_{x\searrow 0} |\log(x)|x^\alpha = 0$.

(b) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where

$$\mu(A) = \sum_{k\in\mathbb{N}} k\mathbf{1}_A(k).$$

By definition

$$\int_{\mathbb{N}} |f|^2 \, d\mu = \sum_{k\in\mathbb{N}} k^2 |f(k)|^2 = \|f\|_V^2,$$

that is, $\|\cdot\|_V$ is a norm associated to an L^2 space. We conclude that V is a Hilbert space.

(c) As in the previous point we have

$$\|f\|_V = \|f\|_{L^1(\mathbb{N},\mathcal{P}(\mathbb{N}),\mu)},$$

where μ is as above. Now, the L^1-norm is not induced by an inner product and hence V endowed with the norm $\|\cdot\|_V$ has no Hilbert space structure. To show this, we check that the parallelogram identity is not satisfied: consider $f = (1, 1, 0, \dots)$ and $g = (1, -1, 0, \dots)$, then

$$\begin{aligned}\|f+g\|_V^2 + \|f-g\|_V^2 &= (2+0)^2 + (0+4)^2 = 20 \\ &\neq 36 = 2\cdot(1+2)^2 + 2\cdot(1+2)^2 = 2\|f\|_V^2 + 2\|g\|_V^2.\end{aligned}$$

(d) Recall the definitions

$$\begin{aligned} C((-\pi,\pi);\mathbb{R}) =& \{\text{ real-valued continuous functions on } (-\pi,\pi)\}, \\ C([-\pi,\pi];\mathbb{R}) =& \{\text{ real-valued continuous functions on } [-\pi,\pi]\}, \\ C_{per}([-\pi,\pi];\mathbb{R}) =& \{f \in C([-\pi,\pi];\mathbb{R}) : f(-\pi) = f(\pi)\}, \\ C^2_{per}([-\pi,\pi];\mathbb{R}) =& \{f \in C([-\pi,\pi];\mathbb{R}) : f \text{ is twice continuously differentiable} \\ & \text{with } f, f', f'' \in C_{per}([-\pi,\pi];\mathbb{R})\}. \end{aligned}$$

From the definitions the following inclusions follow immediatly

$$C^2_{per}([-\pi,\pi];\mathbb{R}) \subset C_{per}([-\pi,\pi];\mathbb{R}) \subset C([-\pi,\pi];\mathbb{R}) \subset C((-\pi,\pi);\mathbb{R}).$$

We claim that all inclusions are strict. For every inclusion we construct a function that belongs to the larger space but not to the smaller one. Let $f, g, h : [-\pi,\pi] \to \mathbb{R}$ be given by

$$\begin{aligned} f(x) &= (\pi - x)^{-1}, \\ g(x) &= x, \\ h(x) &= |x|. \end{aligned}$$

Then

$$\begin{aligned} f &\in C((-\pi,\pi);\mathbb{R}) \backslash C([-\pi,\pi];\mathbb{R}), \\ g &\in C([-\pi,\pi];\mathbb{R}) \backslash C_{per}([-\pi,\pi];\mathbb{R}), \\ h &\in C_{per}([-\pi,\pi];\mathbb{R}) \backslash C^2_{per}([-\pi,\pi];\mathbb{R}). \end{aligned}$$

Exercise 8.3.
Let $u : [a,b] \to \mathbb{C}$ be continuous and piecewise C^1 on a compact interval $[a,b] \subset \mathbb{R}$ with $u(a) = u(b) = 0$.

(a) Show that

$$\int_a^b |u(x)|^2\,dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b |u'(x)|^2\,dx \tag{1}$$

Remark: (1) is known as the Wirtinger inequality.

(b) For which functions does equality hold in (1)?

Solution:

(a) We may assume without loss of generality that $a = 0$. Otherwise consider the shifted function $u(x-a)$. Thus, let u be defined on $[0, b]$. We extend u to an odd function on $[-b, b]$ by reflection (i.e. $\tilde{u}(-x) = -u(x)$). Since $u(0) = 0$, the odd extension is still continuous and piecewise C^1. We then extend u further periodically to obtain a $2b$-periodic function $\tilde{u}$ on $\mathbb{R}$. Once again, since $u(b) = 0$, we have $\tilde{u}$ continuous and piecewise C^1. Note that $\tilde{u}$ has a Fourier series

$$\tilde{u}(x) = \sum_{n\in\mathbb{Z}} c_n(\tilde{u})e^{i\omega nx} \quad \text{with} \quad c_n(\tilde{u}) = \frac{1}{2b}\int_{-b}^{b} \tilde{u}(x)e^{-i\omega nx}\,dx,$$

where we set $\omega = \frac{2\pi}{2b} = \frac{\pi}{b}$. Since $\tilde{u}$ is odd, we have

$$c_0(\tilde{u}) = \frac{1}{2b}\int_{-b}^{b} \tilde{u}(x)\,dx = 0.$$

The Fourier coefficients of the derivative $\tilde{u}'(x)$ satisfy

$$c_n(\tilde{u}) = in\omega \cdot c_n(\tilde{u}), \quad \forall\, n \in \mathbb{Z},$$

see Proposition 2.17 and Remark 2.18 in the lecture notes. It follows from Parseval's identity that

$$\begin{aligned}\int_{-b}^{b} |\tilde{u}'(x)|^2\,dx = 2b\sum_{n\in\mathbb{Z}\setminus\{0\}} |in\omega c_n(\tilde{u})|^2 = 2b\omega^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} n^2|c_n(\tilde{u})|^2 \\ \geq 2b\omega^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n(\tilde{u})|^2 = \omega^2\int_{-b}^{b} |\tilde{u}(x)|^2\,dx.\end{aligned} \tag{2}$$

Since $\tilde{u}$ was the odd extension of u, we have

$$\int_{-b}^{b} |\tilde{u}(x)|^2\,dx = 2\int_0^b |u(x)|^2\,dx, \qquad \int_{-b}^{b} |\tilde{u'}(x)|^2\,dx = 2\int_0^b |u'(x)|^2\,dx.$$

So (1) follows from (2) when we insert $\omega^2 = \frac{\pi^2}{b^2} = \frac{\pi^2}{(b-a)^2}$ (recall $a = 0$).

(b) In (2) we used the inequality $\sum_{n\in\mathbb{Z}} |c_n(\tilde{u})|^2 \leq \sum_{n\in\mathbb{Z}} n^2|c_n(\tilde{u})|^2$ where $c_0(\tilde{u}) = 0$. Equality holds if and only if $c_n(\tilde{u}) = 0$ for all n with $n^2 \neq 1$. Thus, $\tilde{u}$ must have $c_1(\tilde{u})$, $c_{-1}(\tilde{u})$ as its only non-zero Fourier coefficients. Moreover, since $\tilde{u}$ is odd, we must have $c_{-1}(\tilde{u}) = -c_1(\tilde{u})$. Thus, we have equality in (1) if and only if

$$\tilde{u}(x) = c_1(\tilde{u})(e^{i\omega x} - e^{-i\omega x}) = 2ic_1(\tilde{u})\sin(\omega x),$$

i.e.

$$u(x) = C\sin\Big(\frac{\pi}{a-b}x\Big)$$

for some $C \in \mathbb{C}$.

Exercise 8.4.
Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and integrable on $[-\pi, \pi]$.

(a) Show that

$$c_n(f) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \tfrac{\pi}{n}\big) e^{-inx}\, dx,$$

and hence

$$c_n(f) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \Big(f(x) - f\big(x + \tfrac{\pi}{n}\big)\Big) e^{-inx}\, dx.$$

(b) Now assume that f is Hölder continuous of order α, that is

$$|f(x+h) - f(x)| \le C|h|^\alpha$$

for some $0 < \alpha \le 1$, some $C > 0$ and all x, h.
Show that $c_n(f)$ is of order $|n|^{-\alpha}$, i.e. for some $\tilde{C} > 0$ and all $n \in \mathbb{Z}$:

$$|c_n(f)| \le \frac{\tilde{C}}{|n|^\alpha}.$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x},$$

where $0 < \alpha < 1$, is Hölder continuous of order α and satisfies $c_n(f) = n^{-\alpha}$ whenever $n = 2^k$.
Hint: Break the sum up as follows $f(x+h) - f(x) = \sum_{2^k \le |h|^{-1}} \cdots + \sum_{2^k > |h|^{-1}} \ldots$ and use the fact that $|1 - e^{i\theta}| \le |\theta|$ for any $\theta \in \mathbb{R}$.

Solution:

(a) We calculate

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \tfrac{\pi}{n}\big) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y) e^{-iny} e^{i\pi}\, dx = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dx = -c_n(f),$$

where we made the change of variables $y = x - \frac{\pi}{n}$, used the periodicity of f and $e^{i\pi} = -1$. The second equality follows by adding the two different expressions for $c_n(f)$.

(b) Using the previous subquestion and the Hölder condition, we find

$$|c_n(f)| = \Big|\frac{1}{4\pi} \int_{-\pi}^{\pi} \Big(f(x) - f\big(x + \tfrac{\pi}{n}\big)\Big) e^{-inx}\, dx\Big| \le \frac{1}{4\pi} \int_{-\pi}^{\pi} \big|f(x) - f\big(x + \tfrac{\pi}{n}\big)\big|\, dx \le \frac{1}{4\pi} 2\pi C \Big|\frac{\pi}{n}\Big|^\alpha$$

and the inequality follows for $\tilde{C} = \frac{1}{2} C \pi^\alpha$.

(c) We first establish the Hölder condition for f. We estimate

$$\begin{aligned}|f(x+h)-f(x)| &\leq \Big|\sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k(x+h)} - \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}\Big| \leq \sum_{k=0}^{\infty} 2^{-k\alpha}\big|e^{i2^k h}-1\big| \\ &= \sum_{2^k\leq |h|^{-1}} 2^{-k\alpha}\big|e^{i2^k h}-1\big| + \sum_{2^k > |h|^{-1}} 2^{-k\alpha}\big|e^{i2^k h}-1\big| \\ &\leq \sum_{2^k\leq |h|^{-1}} 2^{-k\alpha}\, 2^k|h| + \sum_{2^k>|h|^{-1}} 2\cdot 2^{-k\alpha},\end{aligned}$$

where in first sum we estimated $|e^{i2^k h}-1| \leq 2^k|h|$, and for the second sum we simply used $|e^{i2^k h}-1|\leq 2$. The second sum can now be bounded by

$$\sum_{2^k>|h|^{-1}} 2\cdot 2^{-k\alpha} \leq 2|h|^\alpha \sum_{k=0}^{\infty} 2^{-k\alpha} \leq \frac{2}{1-2^{-\alpha}}|h|^\alpha.$$

For the first sum, let $m\in\mathbb{N}$ be the unique integer such that $2^{-m-1}<|h|\leq 2^{-m}$ (we assume $|h|\leq 1$, otherwise the sum is zero). Then

$$\sum_{2^k\leq |h|^{-1}} 2^{-k\alpha}\, 2^k|h| = \sum_{k=0}^{m} (2^k|h|)^{1-\alpha}|h|^\alpha \leq |h|^\alpha \sum_{k=0}^{m} (2^{k-m})^{1-\alpha} \leq \frac{1}{1-2^{\alpha-1}}|h|^\alpha.$$

This establishes the Hölder continuity of f. Since the series defining f converges in L^2 (in fact uniformly), the Fourier coefficients can be read off from the definition. We have

$$c_n(f) = \begin{cases} n^{-\alpha} & \text{when } n=2^k \text{ for some } k\in\mathbb{N}_0 \\ 0 & \text{otherwise}\end{cases}.$$

Exercise 8.5.

Recall that the Dirichlet kernel satisfies $D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$, for all $n\geq 1$ and $x\in\mathbb{R}$.

(a) Show that

$$\int_0^\pi |D_n(x)|\,dx > 2\sum_{j=0}^{n-1}\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y}.$$

Hint: Use $|\sin(t)|\leq |t|$, then change variables and divide up the domain of integration.

(b) Show that for each $j\geq 0$ we have

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y} \geq \frac{c}{j+1},$$

for some (explicit) constant $c>0$.

(c) Conclude that $\|D_n\|_{L^1(0,\pi)} \geq C \log n$ as $n \to \infty$ for some $C > 0$.
Hint: Recall the asymptotic behavior of the harmonic series: $H_n \coloneqq \sum_{k=1}^{n} \frac{1}{k} \geq \log n$.
Remark: This shows that D_n does not converge in L^1 as $n \to \infty$.

Solution:

(a) In order to show the estimate, we first make the simple observation that

$$\int_0^\pi |D_n(x)| dx = \int_0^\pi \left| \frac{\sin((n+1/2)x)}{\sin(x/2)} \right| dx \geq \int_0^\pi 2 \left| \frac{\sin((n+1/2)x)}{x} \right| dx,$$

since $|\sin(t)| \leq |t|$ for any $t \in \mathbb{R}$. Next we change variables setting $y(x) = \left(n + \frac{1}{2}\right) \cdot x$ to obtain

$$\begin{aligned}\int_0^\pi 2 \left| \frac{\sin((n+1/2)x)}{x} \right| dx &= \int_0^\pi 2 \underbrace{\left(n + \frac{1}{2}\right)}_{=y'(x)} \left| \frac{\sin(y(x))}{y(x)} \right| dx \\ &= 2 \int_0^{(n+1/2)\pi} |\sin(y)| \frac{dy}{y} > 2 \int_0^{n\pi} |\sin(y)| \frac{dy}{y}.\end{aligned}$$

By dividing the domain of the latter integral into intervals of length π and plugging the result into the first estimate above, we directly obtain

$$\int_0^\pi |D_n(x)| dx > 2 \cdot \sum_{j=0}^{n-1} \int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y}.$$

(b) Next we estimate the integral over the subintervals. First note that for any $j \in \mathbb{N}$ we obtain, changing variables $z \coloneqq y - j\pi$,

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y} = \int_0^\pi |\sin(z + j\pi)| \frac{dz}{z + j\pi} = \int_0^\pi |\sin(z)| \frac{dz}{z + j\pi},$$

where we used that $|\sin(z)|$ is π-periodic. We further note that

$$\int_0^\pi |\sin(z)| \frac{dz}{z + j\pi} \geq \int_0^\pi |\sin(z)| \frac{dz}{(1+j)\pi} = \frac{1}{(1+j)\pi} \int_0^\pi |\sin(z)| dz.$$

Now, setting

$$c \coloneqq \frac{1}{\pi} \int_0^\pi |\sin(y)| dy = \frac{1}{\pi} \left[-\cos(y)\right]_0^\pi = \frac{2}{\pi}$$

we obtain

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y} \geq \frac{c}{j+1}.$$

(c) Using part (1) and (2) we obtain

$$\begin{aligned}\|D_n\|_{L^1(0,\pi)} = \int_0^\pi |D_n(x)| dx &> 2 \cdot \sum_{j=0}^{n-1} \int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y} \\ &\geq 2 \cdot \sum_{j=0}^{n-1} \frac{c}{j+1} = 2c \sum_{j=1}^{n} \frac{1}{j} \\ &= 2c \cdot H_n,\end{aligned}$$

where H_n are the partial sums of the harmonic series. Thus,

$$\|D_n\|_{L^1(0,\pi)} \geq 2c\log(n)$$

as $n \to \infty$.