Exercise 9.1. ♣
Which of the following statements are true? Recall that $\hat{f}$ denotes the Fourier transform of a function f.

(a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of compactly supported continuous functions on $\mathbb{R}^d$. If $f_n \to f$ uniformly, then f is continuous and compactly supported.

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of compactly supported continuous functions on $\mathbb{R}^d$. If $f_n \to f$ uniformly, then f is continuous and satisfies $\lim_{|x|\to\infty} f(x) = 0$.

(c) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$.

(d) If f is compactly supported on $\mathbb{R}^d$, then $\hat{f} \in L^1(\mathbb{R}^d)$.

(e) If f is compactly supported and bounded on $\mathbb{R}^d$, then $\hat{f}$ is continuous and satisfies $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

(f) For $f \in L^1(\mathbb{R}^d)$ define $f_t(x) := f(x)\mathbf{1}_{\{|f(x)|\geq t\}}$ for $t > 0$. Then

$$\sup_{\xi\in\mathbb{R}^d} |\hat{f}_t(\xi)| \to 0 \text{ as } t \to \infty.$$

Solution:

(a) False. f is necessarily continuous, see below, but does not necessarily have compact support in $\mathbb{R}^d$. We construct a counterexample. Define $v_n : [0,\infty) \to \mathbb{R}$ by

$$v_n(r) = \begin{cases} 1, & \text{if } r \leq n, \\ n+1-r, & \text{if } n \leq r \leq n+1, \\ 0, & \text{if } r \geq n+1. \end{cases}$$

Let $f_n(x) = v_n(|x|)e^{-|x|^2}$ and $f(x) = e^{-|x|^2}$. Then each f_n has compact support in $\mathbb{R}^d$ and $f_n \to f$ in L^∞ as $n \to \infty$, but f is not compactly supported.

(b) True. Recall that $(C(\mathbb{R}^d), \|\cdot\|_{L^\infty})$ is a Banach space, hence the limit f is certainly continuous. We claim that $f(x) \to 0$ as $|x| \to \infty$. Indeed, for any $\epsilon > 0$ we can find $n \in \mathbb{N}$ such that $\|f_n - f\|_{L^\infty} < \epsilon$. Furthermore, there is $R = R_n > 0$ large enough such that $\operatorname{supp}(f_n) \subset B_R(0)$. Now for all $x \in \mathbb{R}^d$ with $|x| \geq R$ we have

$$|f(x)| = |f(x) - f_n(x)| \leq \|f - f_n\|_{L^\infty} < \epsilon.$$

Thus $|f(x)| \to 0$ as $|x| \to \infty$.

(c) False. Take for example $f = \chi_{[-1,1]} \in L^1(\mathbb{R})$, then

$$\hat{f}(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin\xi}{\xi} \notin L^1(\mathbb{R}).$$

Indeed

$$\int_{\mathbb{R}} \left|\frac{\sin\xi}{\xi}\right| d\xi = 2\sum_{k=0}^{\infty}\int_{k\pi}^{(k+1)\pi} \frac{|\sin\xi|}{|\xi|}d\xi \geq 2\sum_{k=0}^{\infty}\int_{k\pi}^{(k+1)\pi} \frac{|\sin\xi|}{(k+1)\pi}d\xi = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{1+k} = \infty.$$

(d) False. The same counterexample as above still works.

(e) True. If f is compactly supported and bounded, then $f \in L^1(\mathbb{R}^d)$ and the Fourier transform maps L^1 into $\mathcal{C}_0(\mathbb{R}^d)$.

(f) True. Indeed, by the dominated convergence theorem we have $f_t \to 0$ in $L^1(\mathbb{R}^d)$. Note that we can dominate f_t by f itself and the pointwise limit is 0 since $|\{|f| = \infty\}| = 0$ (as f is L^1). By Theorem 3.3 we conclude

$$\|\widehat{f_t}\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\|f_t\|_{L^1(\mathbb{R}^d)} \to 0, \quad \text{as } t \to \infty$$

Exercise 9.2.
For the following PDEs of evolution type, try to find the most general solution of the form $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{-ikx}$ without worrying about convergence issues (i.e. we are looking for 2π-periodic solutions to the PDEs). For each PDE also write down a specific example solution which is not a constant.
Remark: The functions $\{u_k(t)\}_{k\in\mathbb{Z}}$ will of course depend on the initial conditions, in particular the Fourier coefficients of $u(0,\cdot)$ (and sometimes also of $\partial_t u(0,\cdot)$).

(a) $\partial_t u = \cos(t)\partial_{xx} u$

(b) $\partial_{tt} u - \partial_{xx} u = 0$

(c) $\partial_t u = \frac{1}{1+t^2} u + \partial_{xx} u$

(d) $\partial_t u = \partial_{xx} u + 1$

Solution:

(a) We write u as $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$ and notice that

$$\partial_t u = \sum_{k\in\mathbb{Z}} u_k'(t)e^{ikx}, \qquad \cos(t)\partial_{xx} u = \sum_{k\in\mathbb{Z}} -k^2\cos(t)u_k(t)e^{ikx}.$$

Hence we get an ordinary differential equation $u_k'(t) = -k^2\cos(t)u_k(t)$ for each $k \in \mathbb{Z}$. Solving the ODE, we find $u_k(t) = ce^{-k^2\sin(t)}$, where $c = u_k(0)$. So,

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(0)e^{-k^2\sin(t)+ikx}.$$

Example: $u(t,x) = e^{-\sin(t)+ix}$.

(b) We write $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$ and find

$$\partial_{tt} u = \sum_{k\in\mathbb{Z}} u_k''(t)e^{ikx}, \qquad \partial_{xx} u = \sum_{k\in\mathbb{Z}} -k^2 u_k(t)e^{ikx}.$$

So for each $k \in \mathbb{Z}$ we must have $u_k''(t) = -k^2 u_k(t)$. For $k = 0$ this gives $u_0(t) = u_0(0) + u_0'(0)t$, while for $k \neq 0$ we find $u_k(t) = c_1 e^{ikt} + c_2 e^{-ikt}$, where $c_1 = \frac{1}{2}(u_k(0) - \frac{i}{k}u_k'(0))$ and $c_2 = \frac{1}{2}(u_k(0) + \frac{i}{k}u_k'(0))$. So we have

$$u(t,x) = u_0(0) + u_0'(0)t + \sum_{k\in\mathbb{Z}\setminus\{0\}} \Big(\tfrac{1}{2}\big(u_k(0) - \tfrac{i}{k}u_k'(0)\big)e^{ik(t-x)} + \tfrac{1}{2}\big(u_k(0) + \tfrac{i}{k}u_k'(0)\big)e^{-ik(t+x)}\Big).$$

Example: $u(t,x) = \cos(t-x)$.

(c) We write u as $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$ and notice that

$$\partial_t u = \sum_{k\in\mathbb{Z}} u_k'(t)e^{ikx}, \quad \tfrac{1}{1+t^2}u = \sum_{k\in\mathbb{Z}} \tfrac{1}{1+t^2}u_k(t)e^{ikx}, \quad \partial_{xx}u = \sum_{k\in\mathbb{Z}} -k^2 u_k(t)e^{ikx}.$$

Hence we get the ordinary differential equations $u_k'(t) = \frac{1}{1+t^2}u_k(t) - k^2 u_k(t)$ for each $k \in \mathbb{Z}$. This can be integrated to $u_k(t) = ce^{\arctan(t)-k^2t}$, where $c = u_k(0)$. Hence,

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(0)e^{\arctan(t)-k^2t+ikx}.$$

Example: $u(t,x) = e^{\arctan(t)-t+ix}$.

(d) As usual, we write $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$ and observe that u formally satisfies

$$\sum_{k\in Z} u_k'(t)e^{ikx} = \sum_{k\in Z}(-k^2)u_k(t)e^{ikx} + 1.$$

Note that the constant term 1 should be treated with the $k = 0$ Fourier mode. For $k = 0$, we have $u_0' = 1$ and thus $u_0(t) = u_0(0) + t$. For $k \neq 0$, we have

$$u_k(t) = u_k(0)e^{-k^2t}.$$

The general solution is

$$u(t,x) = u_0(0) + t + \sum_{k\in\mathbb{Z}\setminus\{0\}} u_k(0)e^{-k^2t+ikx}.$$

Example: $u(t,x) = t$.

Exercise 9.3.
Consider the following evolution problem with periodic boundary conditions:

$$\begin{cases} i\partial_t u + \partial_{xx}u = 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } 2\pi\text{-periodic } f \in C^\infty(\mathbb{R}). \end{cases}$$

Remark: This PDE is a version of the Schrödinger equation.

(a) Explain why solutions cannot be purely real-valued, unless they are constant.

(b) Explain why, for each fixed large $N \in \mathbb{N}$, we have $\sup_{k\in\mathbb{Z}} |k|^N |c_k(f)| < \infty$.

(c) Find the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on the Fourier coefficients of f.

(d) Show that the formal solution is in fact a true solution and is C^∞ in both variables.
Hint: You need to show that the Fourier coefficients $\{c_k(\partial_t^m \partial_x^n u(t,\cdot))\}$ are summable. This follows from the decay of the $\{c_k(f)\}$.

(e) Show that we found the only possible solution: if v is a solution of the problem which is C^2_{per} in space and C^1 in time, then $u = v$.
Hint: Argue exactly as in the proof of uniqueness for the heat equation.

(f) Find the explicit solution u in the case $f(x) = 2\cos(3x)$.

(g) Does this equation enjoy the "smoothing effect" of the heat equation?
Hint: Observe that the size of u_k and the size of $c_k(f)$ are comparable: do we expect regularization?

Solution:

(a) Suppose we have a real-valued solution u which is not identically constant. Then all derivatives of u also have to be real-valued. This implies that for all (t,x)

$$i\mathbb{R} \ni i\partial_t u(x,t) = -\partial_{xx}u(x,t) \in \mathbb{R} \implies \partial_t u = \partial_{xx} u = 0.$$

So u needs to be constant in time and affine in space, i.e. $u(t,x) = a + bx$. The periodic boundary conditions force $b = 0$, so u is constant.

(b) Since f is 2π-periodic and smooth, we have $f \in C^N_{per}([-\pi,\pi])$ for all $N \in \mathbb{N}$. Thus, Theorem 2.22 (ii) in the lecture notes implies that $\sum_k |k|^\alpha |c_k(f)| < \infty$ for all $\alpha \geq 0$. In particular, $|k|^\alpha |c_k(f)| \to 0$ as $k \to \infty$. Thus, for any fixed $N \in \mathbb{N}$, we have $\sup_k |k|^N |c_k(f)| < \infty$.

(c) If we make the Ansatz $u(t,x) = \sum_k u_k(t)e^{ikx}$ and derive u formally, we get (similar to the case of the heat equation):

$$i\partial_t u(t,x) = \sum_{k\in\mathbb{Z}} iu_k'(t)e^{ikx}, \qquad \partial_{xx}u(t,x) = \sum_{k\in\mathbb{Z}} -k^2 u_k(t)e^{ikx}.$$

Imposing $i\partial_t u + \partial_{xx} u = 0$ and $u(0,x) = f(x)$, the coefficient functions $u_k(t)$ have to solve the following ODE:

$$\begin{cases} u_k'(t) = -ik^2 u_k(t) \\ u_k(0) = c_k(f), \end{cases}$$

which is solved by $u_k(t) = c_k(f)e^{-ik^2 t}$. So we can write our general formal solution to the periodic Schrödinger equation as follows:

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k(f)e^{-ik^2 t}e^{ikx}. \tag{1}$$

(d) We begin first by showing smoothness. Define

$$w_k(t,x) = c_k(f)e^{-ik^2t}e^{ikx}.$$

We show that for $m,n \in \mathbb{N}$, the sums

$$\sum_{k\in\mathbb{Z}} \partial_t^m \partial_x^n w_k(t,x)$$

converge absolutely and uniformly on $\mathbb{R}\times\mathbb{R}$. Note that

$$|\partial_t^m \partial_x^n w_k(t,x)| = |(-ik^2)^m (ik)^n c_k(f)e^{-ik^2t}e^{ikx}| = |k|^{2m+n}|c_k(f)|, \qquad \forall\, t,x \in \mathbb{R}.$$

We now set $N = 2m+n+2$. Thanks to part (b), we know that $\sup_k |k|^N |c_k(f)| < \infty$, i.e. there is some constant $C_N \geq 0$ such that $|k|^N|c_k(f)| \leq C_N$ for all $k \in \mathbb{Z}$.

It follows that

$$\sum_{k\in\mathbb{Z}} \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}} |\partial_t^m \partial_x^n w_k(t,x)| = \sum_{k\in\mathbb{Z}} |k|^{2m+n}|c_k(f)| \leq \sum_{k\in\mathbb{Z}} C_N |k|^{-N}|k|^{2m+n} = C_N \sum_{k\in\mathbb{Z}} |k|^{-2}.$$

Thus, the sum indeed converges absolutely and uniformly and the formally defined function u is actually well-defined. Moreover, the uniform convergence of the sum above implies that we can interchange summation and derivatives, that is:

$$\partial_t^m \partial_x^n u(t,x) = \partial_t^m \partial_x^n \sum_{k\in\mathbb{Z}} w_k(t,x) = \sum_{k\in\mathbb{Z}} \partial_t^m \partial_x^n w_k(t,x)$$

and since this converges absolutely with respect to the uniform norm on $\mathbb{R}\times\mathbb{R}$, the derivative $\partial_t^m \partial_x^n u(t,x)$ exists and is continuous. Since $m,n\in\mathbb{N}$ were arbitrary, this implies that $u \in C^\infty(\mathbb{R}^2)$.

It follows now directly from the remark above about interchanging derivatives and summation that u actually solves the Schrödinger equation:

$$\begin{aligned}
i\partial_t u(t,x) &= i\partial_t \sum_{k\in\mathbb{Z}} c_k(f)e^{-ik^2t}e^{ikx} = \sum_{k\in\mathbb{Z}} i\partial_t c_k(f)e^{-ik^2t}e^{ikx} \\
&= \sum_{k\in\mathbb{Z}} k^2 c_k(f)e^{-ik^2t}e^{ikx} = -\sum_{k\in\mathbb{Z}} \partial_{xx} c_k(f)e^{-ik^2t}e^{ikx} \\
&= -\partial_{xx} \sum_{k\in\mathbb{Z}} c_k(f)e^{-ik^2t}e^{ikx} = -\partial_{xx} u(t,x).
\end{aligned}$$

(e) Let v be as described another solution with the same initial data. Since $v(t,\cdot)$ is in C^2_{per} for each t, we can write

$$v(t,x) = \sum_{k\in\mathbb{Z}} d_k(t)e^{ikx},$$

where $d_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)e^{-ikx}\,dx = c_k(v(t,\cdot))$.

As in the proof for the heat equation, we first show that $d_k(t) \in C^1(\mathbb{R})$ for all k. For this, fix $t \in \mathbb{R}$. We want to show that $\lim_{s\to t} d_k(s) = d_k(t)$. Since v is continuous, it is bounded on the (compact) rectangle $[t-1,t+1]\times[-\pi,\pi]$ by some constant K_t. Thus, for s close enough to t, we can dominate

$v(s,x)e^{-ikx}$ by the constant function K_t. Now, by dominated convergence and continuity of v we get:

$$\lim_{s\to t} d_k(s) = \lim_{s\to t} \frac{1}{2\pi}\int_{-\pi}^{\pi} v(s,x)e^{-ikx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\lim_{s\to t} v(s,x)}_{=v(t,x)}\, e^{-ikx}\,\mathrm{d}x = d_k(t).$$

Since by assumption also $\partial_t v$ is continuous, we can interchange integral and differentiation:

$$d_k'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)e^{-ikx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \partial_t v(t,x)e^{-ikx}\,\mathrm{d}x.$$

Continuity of $d_k'(t)$ is now shown exactly as above, just with v replaced by $\partial_t v$. Thus, $d_k \in C^1(\mathbb{R})$.

Next, define $F_k(t) = e^{ik^2t}d_k(t)$. F_k is differentiable and

$$\begin{aligned}
F_k'(t) &= ik^2e^{ik^2t}d_k(t) + e^{ik^2t}d_k'(t)\\
&= ik^2e^{ik^2t}d_k(t) + e^{ik^2t}c_k(\partial_t v(t,\cdot))\\
&= ik^2e^{ik^2t}d_k(t) + ie^{ik^2t}c_k(\partial_{xx} v(t,\cdot))\\
&= ik^2e^{ik^2t}d_k(t) - ik^2e^{ik^2t}\underbrace{c_k(v(t,\cdot))}_{d_k(t)} = 0,
\end{aligned}$$

for all $t\in\mathbb{R}$. So F_k is a constant function with $F_k(t) = d_k(0) = c_k(v(0,\cdot)) = c_k(f)$. Hence for all $k\in\mathbb{Z}$ and $t\in\mathbb{R}$, we must have $d_k(t) = c_k(f)e^{-ik^2t}$.

We can conclude that

$$v(t,x) = \sum_k d_k(t)e^{ikx} = \sum_k c_k(f)e^{-ik^2t}e^{ikx} = u(t,x),$$

where the equality holds in L^2, so almost everywhere. But both u and v are assumed to be continuous, so equality actually holds everywhere. This shows that (1) has to be the unique solution for the Schrödinger equation.

(f) We can write $f(x)$ as $e^{3ix}+e^{-3ix}$. Inserting this into (1), we get the solution

$$u(t,x) = e^{-9it}\left(e^{3ix}+e^{-3ix}\right) = 2e^{-9it}\cos(3x).$$

(g) Recall that for the heat equation, the solution is smooth for positive times, even if the initial data is not smooth (for instance if it is only C^1_{per}). If v is a solution to the heat equation with initial data f, then $v(t,x) = \sum_k c_k(f)e^{-k^2t}e^{ikx}$. The factor e^{-k^2t} is what gives v its regularity, as it dominates – for $t>0$ – any polynomial in k. If we compare this to a solution u of the Schrödinger equation with the same initial data, i.e. $u(t,x) = \sum_k c_k(f)e^{-ik^2t}e^{ikx}$, we see that we need fast decay of the $c_k(f)$ in order to have regularity, since the factor e^{-ik^2t} has modulus 1 for all k, hence it does not contribute to convergence of the series. (Compare the calculations in part (d), where the fast decay of the $c_k(f)$ was crucial, with the calculations you did in class in the Proof of Theorem 2.34 (iii)).

So in general, the regularity of solutions to the Schrödinger equation at positive times depends on the regularity of the initial data, contrary to the heat equation.

Exercise 9.4.
Given $f \in L^2((-\pi,\pi))$ as initial data, consider the associated periodic solution to the heat equation defined by

$$u(t,x) := \sum_{k\in\mathbb{Z}} c_k(f) e^{ikx-k^2t}, \quad \text{for all } x \in \mathbb{R},\ t > 0.$$

(a) Show that $u \in C^\infty((0,\infty)\times\mathbb{R})$ and u solves the heat equation

$$\partial_t u(t,x) = \partial_{xx} u(t,x), \qquad \text{for all } x \in \mathbb{R},\ t > 0. \tag{2}$$

Hint: Start from Parseval's identity and argue as in the proof of Theorem 2.34.

(b) Show that u assumes the initial datum f in the following L^2 sense:

$$\lim_{t\downarrow 0} \|u(t,\cdot) - f\|_{L^2(-\pi,\pi)} = 0. \tag{3}$$

(c) Consider a function $v(t,x)$ defined in $(0,\infty)\times\mathbb{R}$ which is 2π-periodic and of class C^2 in space, and of class C^1 in time. Show that if v satisfies equations (2) and (3), then $v = u$.

Solution:

(a) As f is in $L^2((-\pi,\pi))$, we make use of Parseval's identity

$$2\pi \sum_{k\in\mathbb{Z}} |c_k(f)|^2 = ||f||_{L^2}^2 < \infty,$$

which implies the simple ℓ^∞ bound

$$\sup_{k\in\mathbb{Z}} |c_k(f)| \le \frac{1}{\sqrt{2\pi}} \|f\|_{L^2}.$$

We consider the candidate solution

$$u(t,x) := \sum_{k\in\mathbb{Z}} c_k(f)\, e^{-k^2t}\, e^{ikx}.$$

We claim that the above series converges absolutely for each (t,x) with $x \in \mathbb{R}$ and $t > 0$, and defines a function of class $C^\infty((0,\infty)\times\mathbb{R})$ which solves the heat equation. To this end, we set $\Omega_\delta := (\delta,\infty)\times\mathbb{R}$ for some arbitrary $\delta > 0$ and, for each $m,n \in \mathbb{N}$, we prove the uniform convergence in Ω_δ of the series of functions

$$\sum_{k\in\mathbb{Z}} \partial_t^m\, \partial_x^n (c_k(f)\, e^{-k^2t}\, e^{ikx}).$$

This will imply that our function u is indeed smooth on Ω_δ. We note that

$$\begin{aligned}\sup_{(x,t)\in\Omega_\delta} \left|\partial_t^m\, \partial_x^n (c_k(f)\, e^{-k^2t}\, e^{ikx})\right| &= \sup_{(x,t)\in\Omega_\delta} \left|(-k^2)^m\, (ik)^n\, c_k(f)\, e^{-k^2t}\, e^{ikx}\right| \\ &\le |k|^{2m+n}\, |c_k(f)|\, e^{-k^2\delta}.\end{aligned}$$

The absolute convergence of the above series with respect to the uniform norm on Ω_δ follows, since

$$\sum_{k\in\mathbb{Z}} \sup_{(x,t)\in\Omega_\delta\times\mathbb{R}} \left|\partial_t^m\,\partial_x^n(c_k(f)\,e^{-k^2t}\,e^{ikx})\right| \leq \sum_{k\in\mathbb{Z}} |k|^{2m+n}\,|c_k(f)|\,e^{-k^2\delta}$$
$$\leq \sup_{k\in\mathbb{Z}} |c_k(f)| \sum_{k\in\mathbb{Z}} |k|^{2m+n}\,e^{-k^2\delta} \leq \|f\|_{L^2}\,C(\delta,m,n) < \infty,$$

where $C(\delta,m,n)$ is a constant depending on m,n and δ. Hence, the potential solution u defines a smooth function in Ω_δ for each $\delta>0$. From the uniform convergence of the derivatives of all order, we also deduce that we can interchange the order of differentiation and the infinite sum, which guarantees that u is a genuine solution in Ω_δ. Since $\delta>0$ was arbitrary, the claim follows.

(b) Again, we apply Parseval's identity and find

$$\lim_{t\to 0} ||u(t,\cdot)-f||_{L^2}^2 = \lim_{t\to 0}\sum_{k\in\mathbb{Z}} \left|c_k(f)\,e^{-k^2t}-c_k(f)\right|^2 = \lim_{t\to 0}\sum_{k\in\mathbb{Z}} |c_k(f)|^2\,\underbrace{|e^{-k^2t}-1|^2}_{\to 0\text{ for fixed }k.} = 0.$$

The last equality is justified by using the Dominated Convergence Theorem (for the sequence of functions $\phi_t(k) := |c_k(f)|^2|e^{-k^2t}-1|^2 \in \ell^1(\mathbb{Z}) = L^1(\mathbb{Z},\mathcal{P}(\mathbb{Z}),\#)$). In fact, the series on the RHS is dominated uniformly in t by

$$|c_k(f)|^2|e^{-k^2t}-1|^2 \leq 4|c_k(f)|^2 \in \ell^1(\mathbb{Z}),$$

and converges pointwise to 0 as $t\to 0$.

(c) Let v be as in the question. We proceed as we did in the lecture notes, i.e, we show that u and v must have identical Fourier coefficients.

Let $v(t,x)=\sum_{k\in\mathbb{Z}} d_k(t)\,e^{ikx}$ be the Fourier series representation of v. The dominated convergence theorem ensures that the Fourier coefficients

$$d_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)\,e^{-ikx}\,dx$$

are C^1 on $(0,\infty)$. Indeed, let $t_0\in(0,\infty)$ and fix some small $\delta>0$. We can dominate $v(t,x)$ by the constant $||v||_{L^\infty([t_0-\delta,t_0+\delta]\times\mathbb{R})}$ for all $t\in[t_0-\delta,t_0+\delta]$ (here we use continuity of v), which is trivially in $L^1((-\pi,\pi))$. Thus, dominated convergence gives

$$\lim_{t\to t_0} d_k(t) = \lim_{t\to t_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)\,e^{-ikx}\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\lim_{t\to t_0} v(t,x)\,e^{-ikx}\,dx = d_k(t_0)$$

proving the continuity of d_k for arbitrary k.

Again, using continuity of $\partial_t v$, we can dominate $\partial_t v(t,x)$ for all $t\in[t_0-\delta,t_0+\delta]$ by the constant $\|\partial_t\, v\|_{L^\infty([t_0-\delta,t_0+\delta]\times\mathbb{R})}$. Applying theorem A.33 (differentiation under the integral) we obtain

$$\frac{d}{dt}d_k(t)\Big|_{t=t_0} = \frac{d}{dt}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)\,e^{-ikx}\,dx\Big|_{t=t_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\partial_t\,v(t,x)\,e^{-ikx}\,dx\Big|_{t=t_0} = c_k(\partial_t\,v(t_0,\cdot)),$$

ensuring the differentiability of $d_k(t)$ on $(0,\infty)$. An analogous argument to the one we used above shows that the d_k are in fact continuously differentiable on the same interval.

Using the continuous differentiability of the d_k, equation (2), and the behaviour of Fourier coefficients under differentiation, we derive

$$\begin{aligned}\frac{d}{dt}\, e^{k^2 t} d_k(t) = e^{k^2 t}\left(k^2 d_k(t) + c_k(\partial_t v(t,\cdot))\right) &= e^{k^2 t}\left(k^2 d_k(t) + c_k(\partial^2_{xx} v(t,\cdot))\right)\\ &= e^{k^2 t}\left(k^2 d_k(t) + (ik)^2 \underbrace{c_k(v(t,\cdot))}_{=d_k(t)}\right) = 0.\end{aligned}$$

Thus, for all $k \in \mathbb{Z}$ we have $d_k(t) = \lambda_k e^{-k^2 t}$ on $(0,\infty)$ for some $\lambda_k \in \mathbb{C}$.

Now we know that both u, v satisfy equation (3), which implies

$$\begin{aligned}\lim_{t\to 0} ||v(t,\cdot) - u(t,\cdot)||_{L^2} &= \lim_{t\to 0} ||v(t,\cdot) - f + f - u(t,\cdot)||_{L^2}\\ &\le \lim_{t\to 0}\big(||v(t,\cdot) - f||_{L^2} + ||u(t,\cdot) - f||_{L^2}\big) = 0.\end{aligned}$$

Thus, by Parseval's theorem, we find

$$\lim_{t\to 0} \sum_{k\in\mathbb{Z}} \left|(\lambda_k - c_k(f))e^{-k^2 t}\right|^2 = 0.$$

In particular, this implies that for each fixed k we have

$$\lim_{t\to 0} |\lambda_k - c_k(f)|^2\, e^{-2k^2 t} = |\lambda_k - c_k(f)|^2 = 0,$$

so we must have $\lambda_k = c_k(f)$, for all $k \in \mathbb{Z}$. This means that u and v have identical Fourier coefficients and are thus equal almost everywhere. But u, v are both continuous on $(0,\infty) \times \mathbb{R}$, so equality almost everywhere implies equality everywhere, completing the proof.

Exercise 9.5.
Consider the following evolution problem with periodic boundary conditions:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u + \lambda u = 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R},\ \text{where } \lambda \ge 0 \text{ is a given constant},\\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R},\\ u(0,x) = f(x) & \text{for some given } 2\pi\text{-periodic } f \in C^\infty(\mathbb{R}),\\ \partial_t u(0,x) = g(x) & \text{for some given } 2\pi\text{-periodic } g \in C^\infty(\mathbb{R}). \end{cases}$$

Remark: This PDE is known as the Klein-Gordon equation. For $\lambda = 0$ it is just the wave equation.

(a) Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t) e^{ikx}$, where the $u_k(t)$ depend on λ and the Fourier coefficients of f and g.
Hint: Recall that λ is non-negative. You will get the equation for a harmonic oscillator.

(b) Show that the formal solution is in fact a true solution and is C^∞ in both variables.

(c) Show that if we just want our solution u to be in $C^2(\mathbb{R}\times\mathbb{R})$, the assumptions on f and g can be relaxed to:

$$\sum_{k\in\mathbb{Z}}\big(|k|^2|c_k(f)|+|k||c_k(g)|\big)<\infty.$$

(d) Assume that $\lambda=0$, i.e. we are considering the wave equation. Show that for each pair of 2π-periodic functions $\phi,\psi\in C^2(\mathbb{R})$ the function $(x,t)\mapsto\phi(x-t)+\psi(x+t)$ solves the wave equation. Explain why this is compatible with what you found in the previous points.

Solution:

(a) If we assume that the solution has the proposed form, then we infer

$$0=(\partial_t^2-\partial_x^2+\lambda)\sum_{k\in\mathbb{Z}}u_k(t)e^{ikx}=\sum_{k\in\mathbb{Z}}(u_k''(t)+(k^2+\lambda)u_k(t))e^{ikx}.$$

As the vectors $(e^{ikx})_{k\in\mathbb{Z}}$ are linearly independent, we find that for any index $k\in\mathbb{Z}$ the following homogeneous second order linear ODE must hold

$$u_k''(t)+(k^2+\lambda)u_k(t)=0\quad\Rightarrow\quad u_k(t)=a_k\cos(\sqrt{k^2+\lambda}\cdot t)+b_k\sin(\sqrt{k^2+\lambda}\cdot t)$$

for some $a_k,b_k\in\mathbb{C}$. Matching these coefficients with the initial conditions, we find that our solution must take the form

$$u(t,x)=\sum_{k\in\mathbb{Z}}u_k(t)e^{ikx},$$
$$u_k(t):=c_k(f)\cos(\sqrt{k^2+\lambda}\cdot t)+\frac{c_k(g)}{\sqrt{k^2+\lambda}}\sin(\sqrt{k^2+\lambda}\cdot t).$$

(b) We argue the exact same way as for the heat equation. Thus, we want to show that applying any derivatives to the terms in the sum defining u results in a series that is absolutely convergent with respect to the supremum norm in $\mathbb{R}\times\mathbb{R}$, i.e. for any non-negative integers $\alpha,\beta\geq 0$ the following series is bounded

$$\sum_{k\in\mathbb{Z}}\left\|\partial_t^\alpha\partial_x^\beta u_k(t)e^{ikx}\right\|_\infty<+\infty. \tag{4}$$

Indeed, as $f,g\in C^\infty_{\text{per}}$ we know from Theorem 2.22 in the lecture notes that for any $N\in\mathbb{N}$ we have

$$\sum_{k\in\mathbb{Z}}|k|^N(|c_k(f)|+|c_k(g)|)<+\infty.$$

One can quickly see that this bound generalizes to

$$\sum_{k\in\mathbb{Z}}\left(k^2+\lambda\right)^{\frac{N}{2}}(|c_k(f)|+|c_k(g)|)<+\infty,\quad\forall\, N\in\mathbb{N}.$$

We are now in a position to prove the convergence in (4):

$$\sum_{k\in\mathbb{Z}}\left\|\partial_t^\alpha\partial_x^\beta u_k(t)e^{ikx}\right\|_\infty\leq\sum_{k\in\mathbb{Z}^*}|c_k(f)|\cdot\left(k^2+\lambda\right)^{\frac{\alpha+\beta}{2}}+|c_k(g)|\cdot\left(k^2+\lambda\right)^{\frac{\alpha+\beta-1}{2}}<+\infty. \tag{5}$$

(c) Under the weaker assumption, we see that (5) still holds for $\alpha + \beta \leq 2$, thus ensuring C^2 regularity.

(d) Using the chain rule, one quickly verifies that every C^2 function of the form $\phi(x-t) + \psi(x+t)$ constitutes a solution to the wave equation.

To show this is compatible with the general solution above for $\lambda = 0$, we rewrite

$$\cos(\sqrt{k^2}t) = \cos(|k|t) = \cos(kx) = \tfrac{1}{2}\big(e^{ikt} + e^{-ikt}\big)$$
$$\tfrac{1}{\sqrt{k^2}}\sin(\sqrt{k^2}t) = \tfrac{1}{|k|}\sin(|k|t) = \tfrac{1}{k}\sin(kx) = \tfrac{1}{2ik}\big(e^{ikt} - e^{-ikt}\big)$$

The general solution can thus be written:

$$\begin{aligned} u(t,x) &= \sum_{k\in\mathbb{Z}} \left(c_k(f)\cos(kt) + \frac{c_k(g)}{k}\sin(kt)\right) e^{ikx} \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{2}\left(c_k(f) + \frac{c_k(g)}{ik}\right) e^{ik(x+t)} + \sum_{k\in\mathbb{Z}} \frac{1}{2}\left(c_k(f) - \frac{c_k(g)}{ik}\right) e^{ik(x-t)} \\ &= \phi(x+t) + \psi(x-t), \end{aligned}$$

where we defined

$$\phi(y) := \sum_{k\in\mathbb{Z}} \frac{1}{2}\left(c_k(f) + \frac{c_k(g)}{ik}\right)\exp(iky), \quad \psi(y) := \sum_{k\in\mathbb{Z}} \frac{1}{2}\left(c_k(f) - \frac{c_k(g)}{ik}\right)\exp(iky).$$

Note that the series are convergent in $C^2_{per}(\mathbb{R})$ under the decay assumptions on $c_k(f), c_k(g)$.