Exercise 10.1. ♣
Which of the following statements are true?

(a) If $f \in C^m(\mathbb{R})$ and for all derivatives up to order m, we have $f, f', \ldots, f^{(m)} \in L^1(\mathbb{R})$, then

$$\lim_{|\xi|\to\infty} |\xi|^m \hat{f}(\xi) = 0.$$

(b) Let $f \in L^1(\mathbb{R}^d)$ and let A be an invertible $d \times d$ matrix. Then we have

$$\widehat{f \circ A}(\xi) = \hat{f}(A^{-1}\xi).$$

(c) For $f, g \in \mathcal{S}(\mathbb{R})$, the Fourier transform of $h(x,y) = f(2x)g(y/2)$ is $\hat{h}(\xi,\eta) = \hat{f}(\xi/2)\hat{g}(2\eta)$.

(d) Let $\psi \in C_c^1(\mathbb{R}^d)$ with $\psi(x) \equiv 1$ in a neighbourhood of $x = 0$. Then for each $f \in L^1(\mathbb{R}^d)$:

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f(x)\psi(\epsilon x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx \quad \text{and} \quad \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\epsilon x)\,dx = 0.$$

Solution:

(a) True. $f \in C^m(\mathbb{R})$ with all derivatives up to m in $L^1(\mathbb{R})$ implies that $\mathcal{F}(f^{(k)})(\xi) = (i\xi)^k \hat{f}(\xi)$ for all $k \leq m$. Thus, in particular $|\xi|^m|\hat{f}(\xi)| = \mathcal{F}(f^{(m)})(\xi)$. Since $f^{(m)} \in L^1(\mathbb{R})$ its Fourier transform is in $C_0(\mathbb{R})$, i.e. goes to zero at infinity, and the claim follows.

(b) False. In fact, using a change of variables $y = Ax$, we have

$$\begin{aligned}\mathcal{F}(f\circ A)(\xi) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(Ax)e^{-ix\cdot\xi}\,dx = \frac{1}{(2\pi)^{\frac{d}{2}}}\frac{1}{|\det(A)|}\int_{\mathbb{R}^d} f(y)e^{-iA^{-1}y\cdot\xi}\,dy \\ &= \frac{1}{|\det(A)|}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(y)e^{-iy\cdot(A^{-1})^\top\xi}\,dy = \frac{1}{|\det(A)|}\mathcal{F}(f)\big((A^{-1})^\top\xi\big),\end{aligned}$$

where $(A^{-1})^\top$ denotes the transpose of the inverse of A.

(c) True. Given $f, g \in \mathcal{S}(\mathbb{R})$, set $h(u,v) = f(2x)g(y/2)$. Then,

$$\begin{aligned}\hat{h}(\xi,\eta) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(2x)g(y/2)e^{-i\xi x - i\eta y}\,dxdy \\ &= 2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\xi x/2}\,dx\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(y)e^{-i2\eta y}\,dy = \hat{f}(\xi/2)\hat{g}(2\eta),\end{aligned}$$

where the second equality follows from Fubini's theorem, since f, g Schwartz implies that h is Schwartz and hence the integral is absolutely convergent.

(d) True. Consider the family of functions $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}$, given by $f_\varepsilon(x) = f(x)\psi(\varepsilon x)$. We know that $\psi \equiv 1$ in $B_r(0)$ for some $r > 0$. Then for any $x \in B_{r/\varepsilon}(0)$, we have εx in $B_r(0)$ and so $f_\varepsilon(x) = f(x)\psi(\varepsilon x) = f(x)$. This shows that $f_\varepsilon \to f$ pointwise as $\varepsilon \to 0$. Since $\psi \in C_c^1(\mathbb{R}^d)$, it is bounded and $|f_\varepsilon(x)| \leq ||\psi||_{L^\infty}|f(x)|$. So $||\psi||_{L^\infty}|f|$ is integrable and bounds f_ε pointwise. We can thus apply the dominated convergence theorem in $L^1(\mathbb{R}^d)$ and get

$$\int_{\mathbb{R}^d} f(x)\psi(\varepsilon x)\,dx = \int_{\mathbb{R}^d} f_\varepsilon(x)\,dx \to \int_{\mathbb{R}^d} f(x)\,dx \text{ as } \varepsilon \to 0.$$

For the second limit, notice $\psi \in C_c^1(\mathbb{R}^d)$ implies that also its derivatives are compactly supported, i.e. $\partial_{x_j}\psi \in C_c^0(\mathbb{R}^d)$ and so they are bounded as well. Moreover, since ψ is constant in a neighbourhood of 0, there is some $r > 0$ such that $\partial_{x_j}\psi \equiv 0$ in $B_r(0)$. Thus by the same argument as above, the family $g_\varepsilon(x) = f(x)\partial_{x_j}\psi(\varepsilon x)$ converges pointwise to 0 as $\varepsilon \to 0$. By boundedness, $|g_\varepsilon(x)| \leq ||\partial_{x_j}\psi||_{L^\infty}|f(x)|$, where the upper bound is again integrable. So again by dominated convergence in $L^1(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\varepsilon x)\,dx = \int_{\mathbb{R}^d} g_\varepsilon(x)\,dx \to 0 \text{ as } \varepsilon \to 0.$$

Exercise 10.2.
Let $f \in L^1(\mathbb{R})$ be a continuous function with Fourier transform

$$\hat{f}(\xi) = \frac{\log(1+\xi^2)}{\xi^2}.$$

Compute the following:

(a) $\int_{\mathbb{R}} f(x)\,dx$,

(b) $f(0)$.

Solution:

(a) By definition, we have $\int_{\mathbb{R}} f(x)\,dx = \sqrt{2\pi}\hat{f}(0)$. Although we cannot directly evaluate the expression for $\hat{f}$ at ξ, the Fourier transform of the L^1 function f must be continuous, so

$$\int_{\mathbb{R}} f(x)\,dx = \sqrt{2\pi}\lim_{\xi\to 0}\hat{f}(\xi) = \sqrt{2\pi}\lim_{\xi\to 0}\frac{\log(1+\xi^2)}{\xi^2} = \sqrt{2\pi}\lim_{\xi\to 0}\frac{\frac{2\xi}{1+\xi^2}}{2\xi} = \sqrt{2\pi},$$

where we used l'Hôpital's rule.

(b) We notice that $\hat{f} \in L^1(\mathbb{R})$. Since both f and $\hat{f}$ are L^1 functions, the Fourier inversion formula holds and we must in particular have

$$\begin{aligned} f(0) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{\log(1+\xi^2)}{\xi^2}\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\log(1+\xi^2)\frac{d}{d\xi}\Big(-\frac{1}{\xi}\Big)\,d\xi = \frac{2}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{1}{1+\xi^2}\,d\xi = \sqrt{\frac{2}{\pi}}\arctan(\xi)\big|_{-\infty}^{\infty} = \sqrt{2\pi}, \end{aligned}$$

where we used partial integration and the fact that $\lim_{\xi\to\pm\infty}\frac{\log(1+\xi^2)}{\xi} = 0$.

Exercise 10.3.

(a) Compute the Fourier transform of $f(x) = e^{-ax^2}$ for $a > 0$.

(b) Compute the convolution $e^{-ax^2} * e^{-bx^2}$ for $a, b > 0$ by using the Fourier transform.

Solution:

(a) We compute

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ax^2 - ix\xi}\, dx.$$

Completing the square gives: $-ax^2 - ix\xi = -a(x + i\frac{\xi}{2a})^2 - \frac{\xi^2}{4a}$. Thus,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a(x+i\frac{\xi}{2a})^2 - \frac{\xi^2}{4a}}\, dx = e^{-\frac{\xi^2}{4a}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a(x+i\frac{\xi}{2a})^2}\, dx.$$

We can interpret the final integral, for fixed $\xi \in \mathbb{R}$, as a contour integral in the complex plane:

$$\int_{\mathbb{R}} e^{-a(x+i\frac{\xi}{2a})^2}\, dx = \int_{\mathbb{R}+i\frac{\xi}{2a}} e^{-az^2}\, dz.$$

Since the integrand is analytic in z and decays exponentially along any line in $\mathbb{C}$ with fixed imaginary part, we can use the Cauchy integral theorem to shift the integration contour from $\mathbb{R} + i\frac{\xi}{2a}$ to $\mathbb{R}$. Thus,

$$\int_{\mathbb{R}+i\frac{\xi}{2a}} e^{-az^2}\, dz = \int_{\mathbb{R}} e^{-az^2}\, dz = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} e^{-z^2}\, dz = \sqrt{\frac{\pi}{a}},$$

where we used the value of the standard Gaussian integral. We have found that

$$\hat{f}(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}$$

(b) Note that $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$ are Schwartz functions when $a, b > 0$. Since the convolution of two Schwartz functions is also a Schwartz function, we have $f * g \in \mathcal{S}(\mathbb{R})$. Thus, we can apply the Fourier inversion formula to this function. Denoting by $\mathcal{F}$, $\mathcal{F}^{-1}$ the Fourier transform and its inverse and recalling that the Fourier transform of a convolution is the product of Fourier transforms, i.e. $\mathcal{F}(f * g) = \sqrt{2\pi}\, \hat{f} \cdot \hat{g}$, we find

$$f * g(x) = \mathcal{F}^{-1} \circ \mathcal{F}(f * g)(x) = \sqrt{2\pi}\, \mathcal{F}^{-1}\big(\hat{f} \cdot \hat{g}\big)(x).$$

Using the formula we found above for the Fourier transform of a Gaussian, we find

$$\hat{f}(\xi)\hat{g}(\xi) = \frac{1}{\sqrt{4ab}} e^{-\frac{\xi^2}{4a} - \frac{\xi^2}{4b}} = \frac{1}{\sqrt{4ab}} e^{-\frac{\xi^2}{4}(\frac{1}{a} + \frac{1}{b})} = \sqrt{\frac{c}{2ab}} \frac{1}{\sqrt{2c}} e^{-\frac{\xi^2}{4c}},$$

where we defined $c = (\frac{1}{a} + \frac{1}{b})^{-1} = \frac{ab}{a+b}$. We recognize the Fourier transform of the function $h(x) = e^{-cx^2}$. Thus, applying the inverse Fourier transform we find

$$f * g(x) = \sqrt{\frac{\pi c}{ab}} \mathcal{F}^{-1}\big(\hat{h}\big)(x) = \sqrt{\frac{\pi c}{ab}} h(x) = \sqrt{\frac{\pi c}{ab}} e^{-cx^2} = \sqrt{\frac{\pi}{a+b}} e^{-\frac{ab}{a+b}x^2},$$

where in the final step we inserted the definition of c.

Exercise 10.4.
Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Show that the sum $\sum_{n\in\mathbb{Z}} f(\sqrt{2\pi}n)$ is convergent and prove the Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f\big(\sqrt{2\pi}n\big) = \sum_{n\in\mathbb{Z}} \hat{f}\big(\sqrt{2\pi}n\big).$$

Hint: Consider the $\sqrt{2\pi}$-periodic function defined by $F(x) = \sum_{n\in\mathbb{Z}} f(x + \sqrt{2\pi}n)$.

Solution: Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \sum_{n\in\mathbb{Z}} f(x + \sqrt{2\pi}n)$. Since $f \in \mathcal{S}(\mathbb{R})$, we in particular have $|f(x+n)| \le C(\sqrt{2\pi} + |x+n|)^{-2}$ for some C independent of x and n. (This follows from

$$\sup_{x\in\mathbb{R}}(\sqrt{2\pi} + |x|)^2|f(x)| \le 2\pi \sup_{x\in\mathbb{R}}|f(x)| + 2\sqrt{2\pi}|x|\sup_{x\in\mathbb{R}}|f(x)| + \sup_{x\in\mathbb{R}}|x|^2|f(x)| \le C$$

for a Schwartz function f.) Thus, for each fixed $x \in \mathbb{R}$, the sum defining $F(x)$ converges absolutely. Since

$$F(x+\sqrt{2\pi}) = \sum_{n\in\mathbb{Z}} f(x + \sqrt{2\pi}(n+1)) = \sum_{n\in\mathbb{Z}} f(x+\sqrt{2\pi}n) = F(x)$$

for every $x \in \mathbb{R}$, we see that F is a periodic function with period $\sqrt{2\pi}$. Moreover, using

$$\sup_{x\in[-\sqrt{\pi},\sqrt{\pi}]} |f(x+\sqrt{2\pi}n)| \le \sup_{x\in[-\sqrt{\pi},\sqrt{\pi}]} \frac{C}{(\sqrt{2\pi}+|x+\sqrt{2\pi}n|)^2} \le \frac{C}{(\sqrt{\pi}+\sqrt{2\pi}|n|)^2},$$

we find that the sum defining F converges uniformly on $[0, \sqrt{2\pi}]$, so F is a continuous $\sqrt{2\pi}$-periodic function. We can thus expand F in terms of its Fourier coefficients

$$\begin{aligned}
c_k(F) &= \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} f(x+\sqrt{2\pi}n)e^{-i\sqrt{2\pi}kx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \int_0^{\sqrt{2\pi}} f(x+\sqrt{2\pi}n)e^{-i\sqrt{2\pi}kx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \int_{\sqrt{2\pi}n}^{\sqrt{2\pi}(n+1)} f(x)e^{-i\sqrt{2\pi}kx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\sqrt{2\pi}kx}\,dx = \hat{f}(\sqrt{2\pi}k).
\end{aligned}$$

A similar argument as above, but estimating the derivative of the sum defining F, shows that F is actually in $C^1(\mathbb{R})$. Thus, the Fourier series for F converges uniformly; in particular, evaluating the Fourier series at $x = 0$ we find

$$F(0) = \sum_{n\in\mathbb{Z}} f(\sqrt{2\pi}n) = \sum_{n\in\mathbb{Z}} c_n(F) = \sum_{n\in\mathbb{Z}} \hat{f}(\sqrt{2\pi}n)$$

Exercise 10.5.
The goal of this problem is to show the existence of the harmonic extension to the interior of the unit disk of a sufficiently regular function f defined on the disk's boundary.

Consider the following second order differential operators in two variables (x_1, x_2):

$$\Delta := \partial_{11} + \partial_{22} \qquad \text{and} \qquad L := \partial_{11} + \frac{1}{x_1}\partial_1 + \frac{1}{x_1^2}\partial_{22}.$$

We say that a twice differentiable function $w(x_1, x_2)$ is harmonic if $\Delta w = 0$ in its domain.

(a) Let $D := \{(x, y) : x^2 + y^2 < 1\}$ be the unit disk. Given $u\colon \overline{D} \to \mathbb{R}$, consider the function[1]

$$v(r, \theta) := u(r\cos\theta, r\sin\theta), \quad r \in [0, 1],\, \theta \in \mathbb{R}. \tag{1}$$

Using the chain rule, check that

$$(\Delta u)(r\cos\theta, r\sin\theta) = Lv(r, \theta), \quad \forall\, r \in (0, 1),\, \theta \in \mathbb{R}.$$

(b) Given any sufficiently regular function $F\colon \partial D \to \mathbb{R}$ consider its 2π-periodic version $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(\theta) := F(\cos\theta, \sin\theta), \quad \theta \in \mathbb{R}.$$

Show that we can find a solution $u\colon \overline{D} \setminus \{0\} \to \mathbb{R}$ of

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \{0\}, \\ u = F & \text{on } \partial D, \end{cases}$$

by instead solving

$$\begin{cases} \partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & \text{in } (0, 1) \times \mathbb{R}, \\ v(r, \theta + 2\pi) = v(r, \theta) & \text{in } (0, 1] \times \mathbb{R}, \\ v(1, \theta) = f(\theta) & \text{for all } \theta \in \mathbb{R}, \end{cases} \tag{2}$$

and then defining u using (1).

(c) Formally solve the system (2) by using the ansatz $v(r, \theta) := \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\theta}$. Explain why the $\{u_k(r)\}$ are not *uniquely* determined by the Fourier coefficients $\{c_k(f)\}$. Explain why they are unique if we further require that

$$\limsup_{r\downarrow 0} |u_k(r)| < \infty \quad \forall k \in \mathbb{Z}. \tag{3}$$

(d) Let $v(r, \theta)$ be the ansatz constructed in the previous subquestion by requiring the extra condition (3). Show that v is of class C^∞ for $(r, \theta) \in (0, 1) \times \mathbb{R}$, as soon as $f \in L^2((-\pi, \pi))$.

(e) ★ Show that, when $f \in L^2((-\pi, \pi))$, the v you constructed with the extra condition (3) in fact corresponds to a u that is of class C^∞ in the whole open disk (including the origin!). Furthermore, this u satisfies $\Delta u = 0$ in D and meets the boundary condition in the sense that

$$\lim_{r\uparrow 1} \big\| u(r\cos(\cdot), r\sin(\cdot)) - F(\cos(\cdot), \sin(\cdot)) \big\|_{L^2((-\pi,\pi))} = 0.$$

[1] This is u in polar coordinates.

Solution:

(a) It is convenient to define $\Phi(r,\theta) = (r\cdot\cos\theta, r\cdot\sin\theta)$. Then $v = u\circ\Phi$. Further, we notice that

$$\begin{cases} \frac{\partial\Phi_1}{\partial r} = \cos\theta & \frac{\partial\Phi_1}{\partial\theta} = -r\sin\theta \\ \frac{\partial\Phi_2}{\partial r} = \sin\theta & \frac{\partial\Phi_2}{\partial\theta} = r\cos\theta. \end{cases}$$

Using this and the chain rule, we obtain

$$\begin{aligned} \partial_r v &= \partial_1 u\cdot\frac{\partial\Phi_1}{\partial r} + \partial_2 u\cdot\frac{\partial\Phi_2}{\partial r} \\ &= \cos\theta\cdot(\partial_1 u)\circ\Phi + \sin\theta\cdot(\partial_2 u)\circ\Phi. \\ \partial_\theta v &= \partial_1 u\cdot\frac{\partial\Phi_1}{\partial\theta} + \partial_2 u\cdot\frac{\partial\Phi_2}{\partial\theta} \\ &= r\cos\theta\cdot(\partial_2 u)\circ\Phi - r\sin\theta\cdot(\partial_1 u)\circ\Phi. \end{aligned}$$

Another application of the chain rule gives us:

$$\begin{aligned} \partial_{rr}v =& \partial_r(\partial_r v) \\ =& \cos\theta\left(\partial_{11}u\cdot\frac{\partial\Phi_1}{\partial r} + \partial_{12}u\cdot\frac{\partial\Phi_2}{\partial r}\right) \\ &+ \sin\theta\cdot\left(\partial_{21}u\cdot\frac{\partial\Phi_1}{\partial r} + \partial_{22}u\cdot\frac{\partial\Phi_2}{\partial r}\right) \\ =& \cos^2\theta\cdot(\partial_{11}u)\circ\Phi + 2\cos\theta\sin\theta\cdot(\partial_{12}u)\circ\Phi + \sin^2\theta\cdot(\partial_{22}u)\circ\Phi. \end{aligned}$$

We can compute $\partial_{\theta\theta}v$ by the same method:

$$\begin{aligned} \partial_{\theta\theta}v =& \partial_\theta(\partial_\theta v) \\ =& \partial_\theta\left(r\cos\theta\cdot(\partial_2 u)\circ\Phi - r\sin\theta\cdot(\partial_1 u)\circ\Phi\right) \\ =& - r\sin\theta\cdot(\partial_2 u)\circ\Phi + r\cos\theta\left(\partial_{21}u\cdot\frac{\partial\Phi_1}{\partial\theta} + \partial_{22}u\cdot\frac{\partial\Phi_2}{\partial\theta}\right) \\ &- r\cos\theta\cdot(\partial_1 u)\circ\Phi - r\sin\theta\cdot\left(\partial_{11}u\cdot\frac{\partial\Phi_1}{\partial\theta} + \partial_{12}u\cdot\frac{\partial\Phi_2}{\partial\theta}\right) \\ =& - r\left(\cos\theta\cdot\partial_1 u\circ\Phi + \sin\theta\cdot\partial_2 u\circ\Phi\right) \\ &+ r^2\left(\sin^2\theta\cdot\partial_{11}u\circ\Phi - 2\cos\theta\sin\theta\cdot\partial_{12}u\circ\Phi + \cos^2\theta\cdot\partial_{22}u\circ\Phi\right) \\ =& - r\partial_r v + r^2\left(\sin^2\theta\cdot(\partial_{11}u)\circ\Phi - 2\cos\theta\sin\theta\cdot(\partial_{12}u)\circ\Phi + \cos^2\theta\cdot(\partial_{22}u)\circ\Phi\right) \end{aligned}$$

Note that this implies (by using $\cos^2\theta + \sin^2\theta = 1$) that

$$\partial_{rr}v + \frac{1}{r^2}\partial_{\theta\theta}v = \partial_{11}u\circ\Phi + \partial_{22}u\circ\Phi - \frac{1}{r}\partial_r v.$$

The claim follows:

$$\begin{aligned} (\Delta u)(r\cdot\cos\theta, r\cdot\sin\theta) &= (\partial_{11}u)\circ\Phi + (\partial_{22}u)\circ\Phi \\ &= \partial_{rr}v + \frac{1}{r}\partial_r v + \frac{1}{r^2}\partial_{\theta\theta}v = Lv(r,\theta). \end{aligned}$$

(b) Assume that $v : (0,1] \times \mathbb{R} \to \mathbb{R}$ is a solution to

$$\begin{cases} \partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & (r,\theta) \in (0,1] \times \mathbb{R} \\ v(r,\theta + 2\pi) = v(r,\theta) & (r,\theta) \in (0,1] \times \mathbb{R} \\ v(1,\theta) = f(\theta) & \theta \in \mathbb{R}. \end{cases}$$

From the periodicity of v with respect to θ, it follows that there must exist $u \colon \overline{D} \setminus \{0\} \to \mathbb{R}$ such that $v(r,\theta) = u(r\cos\theta, r\sin\theta)$. By the first part of the exercise, we have that

$$\begin{aligned} \Delta u = Lv &= \partial_{rr} v + r^{-1}\partial_r v + r^{-2}\partial_{\theta\theta} v \\ &= r^{-2} \cdot (\partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v) = 0, \end{aligned}$$

on $D \setminus \{0\}$ which corresponds to considering $(r,\theta) \in (0,1) \times \mathbb{R}$. The boundary condition can also be checked directly: Let $x \in \partial D$, then there is $\theta \in \mathbb{R}$ such that $x = (\cos\theta, \sin\theta)$. This implies

$$u(x) = v(1,\theta) = f(\theta) = F(\cos\theta, \sin\theta) = F(x).$$

(c) We now want to solve the PDE from part (2) formally. For this we assume

$$v := \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\theta}.$$

We then have

$$\begin{cases} \partial_{\theta\theta} v = -\sum_{k\in\mathbb{Z}} k^2 u_k(r) e^{ik\theta} \\ \partial_r v = \sum_{k\in\mathbb{Z}} u_k'(r) e^{ik\theta} \\ \partial_{rr} v = \sum_{k\in\mathbb{Z}} u_k''(r) e^{ik\theta}. \end{cases}$$

Thus, the PDE $\partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0$ becomes the following system of ODEs:

$$k^2 u_k(r) = r u_k'(r) + r^2 u_k''(r), \qquad \text{for all } k \in \mathbb{Z}.$$

We first consider the case $k = 0$, where we have $r \cdot u_0'(r) + r^2 \cdot u_0''(r) = 0$. This ODE is solved by $u_0(r) = c_0 + d_0 \cdot \log r$, for some arbitrary constants c_0, d_0. For $k \neq 0$ we have $u_k(r) = c_k r^{|k|} + d_k r^{-|k|}$, for arbitrary constants c_k and d_k.

For those unfamiliar with this type of ODE, one way to solve it is to realize that the ODE for u_k above is equivalent to

$$(r\partial_r)^2 u_k(r) = k^2 u_k(r), \quad \text{for } r \in (0,1).$$

We can then make a change of variable $r = e^t$ with $t \in (-\infty, 0)$ and we have $r\partial_r = \partial_t$. Thus, writing $\tilde{u}_k(t) = u_k(e^t)$, we find that $\tilde{u}_k$ must satisfy $\tilde{u}_k''(t) = k^2 \tilde{u}_k(t)$, a simple constant coefficient ODE. One immediately solves: $\tilde{u}_0(t) = c_0 + d_0 t$ and $\tilde{u}_k(t) = c_k e^{|k|t} + d_k e^{-|k|t}$ for $k \neq 0$. Rewriting this in terms of the coordinate $r = \log(t)$ leads to the solutions found above.

We can see that the initial datum $v(1,\theta) = f(\theta)$ is not enough to determine the coefficients, as it only tells us that $u_k(1)$ has to agree with the Fourier coefficient $c_k(f)$, and since the u_k are determined by a second order ODE, this is not sufficient for uniqueness: there is an additional degree of freedom. If we additionally assume,

$$\limsup_{r\downarrow 0} |u_k(r)| < \infty$$

for all $k \in \mathbb{Z}$, this forces the coefficients of the log and of the negative powers of r to be zero. Thus, we obtain $d_k = 0$ for all $k \in \mathbb{Z}$. And matching the condition $u_k(1) = c_k(f)$ we find: $c_k = c_k(f)$ for all $k \in \mathbb{Z}$. We obtain

$$v(r,\theta) = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|} e^{ik\theta}.$$

(d) We now want to show that the solution constructed in the last part is smooth if $f \in L^2$. We know that r^k decays faster than any power of k when $|r| < 1$. Consider now $\Omega_\varepsilon := [0, 1-\varepsilon) \times \mathbb{R}$. We show that $v|_{\Omega_\varepsilon}$ is smooth for any $\varepsilon > 0$. For this, let $\alpha, \beta \in \mathbb{N}$ be arbitrary. Then on Ω_ε we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \|\partial_r^\alpha \partial_\theta^\beta u_k(r) e^{ik\theta}\|_{L^\infty(\Omega_\varepsilon)} &= \sum_{k\in\mathbb{Z}} \left\|\partial_r^\alpha \partial_\theta^\beta (c_k(f) r^{|k|} e^{ik\theta})\right\|_{L^\infty(\Omega_\varepsilon)} \\
&\le \sum_{k\in\mathbb{Z}} |c_k(f)| |k|^\beta \left\|\partial_r^\alpha r^{|k|}\right\|_{L^\infty(\Omega_\varepsilon)} \\
&\le \sum_{k\in\mathbb{Z}} |c_k(f)| |k|^{\beta+1} (|k|-1)\dots(|k|-\alpha+1)(1-\varepsilon)^{|k|-\alpha} < \infty,
\end{aligned}$$

where we used that $\{c_k(f)\} \in \ell^2 \subset \ell^\infty$. Thus, the partial sums and all their derivatives converge uniformly on Ω_ε, which implies $v|_{\Omega_\varepsilon} \in C^\infty(\Omega_\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we have that v is smooth on $[0,1) \times \mathbb{R}$.

(e) We have found a solution $v \in C^\infty((0,1)\times\mathbb{R})$ (with, moreover, all derivatives bounded as $r \to 0$) of the equation $Lv = 0$, given by

$$v(r,\theta) = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|} e^{ik\theta}.$$

We now want to see that v corresponds to a solution $u \in C^\infty(D)$ of the Laplace equation. The relationship between u and v is given by $v(r,\theta) = u(r\cos\theta, r\sin\theta)$ (recall that v is 2π-periodic in θ). This can also be written $v = u \circ \Phi$, where $\Phi(r,\theta) = (r\cdot\cos\theta, r\cdot\sin\theta)$. For $r \neq 0$ the map Φ this is a local diffeomorphism (by the inverse function theorem). Thus, we find that u can be written around any $(x,y) \in D\setminus\{0\}$ as the composition of smooth functions, namely of v and a local inverse of Φ. Thus, u is smooth on $D \setminus \{0\}$. However, we will still have to show smoothness at the origin, since the change of variables is singular there.

By the first two parts of this exercise, u satisfies $\Delta u = 0$ in $D \setminus \{0\}$. Note that if we can show $u \in C^2(D)$, we automatically obtain that u is a solution of $\Delta u = 0$ in the whole D, by continuity.

We next show that u fulfills the boundary condition in the sense that

$$\lim_{r\uparrow 1} \left\|u(r\cos(\cdot), r\sin(\cdot)) - F(\cos(\cdot),\sin(\cdot))\right\|_{L^2((-\pi,\pi))} = 0.$$

Note that, by definition, this is equivalent to

$$\lim_{r\uparrow 1} \left\|v(r,\cdot) - f(\cdot)\right\|_{L^2((-\pi,\pi))} = 0.$$

Now the difference $v(r,\cdot) - f(\cdot)$ certainly lies in $L^2((-\pi,\pi))$ for all $r \in (0,1)$ (since v is smooth and f is L^2) and the Fourier coefficient satisfies

$$c_k\big(v(r,\cdot) - f(\cdot)\big) = c_k(f) r^k - c_k(f), \quad \forall\, k \in \mathbb{Z}.$$

We see directly, that $c_k(v(r,\cdot)-f) \to 0$ as $r \uparrow 1$, pointwise in k. Thus, $\lim_{r\uparrow 1} \|v(r,\cdot)-f(\cdot)\|_{L^2} = 0$ by Parseval's identity and dominated convergence with dominant

$$|c_k(f)|^2(1-r^k)^2 \leq |c_k(f)|^2 \in \ell^1(\mathbb{Z}).$$

We remark that if we have stronger decay conditions on $c_k(f)$ (i.e., f is more regular) then the boundary datum is achieved in stronger norms. For example, if $\sum_k |c_k(f)| < \infty$, then the same computation shows

$$\lim_{r\uparrow 1} \|v(r,\cdot)-f(\cdot)\|_{L^\infty(-\pi,\pi)} = 0,$$

which means that v can be continuously extended to f on ∂D.

We now turn to the smooth extension of u to the origin. Notice that $\lim_{r\to 0} v(r,\theta) = c_0(f)$ is independent of θ. Thus, u certainly has a continuous extension to 0. Similarly, we can express

$$\begin{aligned}(\partial_1 u)\circ\Phi &= \cos\theta\,\partial_r v - \frac{1}{r}\sin\theta\,\partial_\theta v = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|-1}(|k|\cos\theta - ik\sin\theta)e^{ik\theta} \\ &= c_1(f)(\cos\theta - i\sin\theta)e^{i\theta} + c_{-1}(f)(\cos\theta + i\sin\theta)e^{-i\theta} + r\tilde{v}_1(r,\theta) \\ &= c_1(f) + c_{-1}(f) + r\tilde{v}_1(r,\theta),\end{aligned}$$

where $\tilde{v}_1(r,\theta) = \sum_{|k|\geq 2} c_k(f) r^{|k|-2}(|k|\cos\theta - ik\sin\theta)e^{ik\theta}$ remains bounded as $r \to 0$. Since the dependence on θ is only in $\tilde{v}_1$, which is in turn multiplied by r, we have a θ independent limit $\partial_1 u \circ \Phi \to c_1(f) + c_{-1}(f)$ as $r \to 0$. Thus, $\partial_1 u$ extends continuously to the origin. Let us check that the same happens for $\partial_2 u$:

$$\begin{aligned}(\partial_2 u)\circ\Phi &= \sin\theta\,\partial_r v + \frac{1}{r}\cos\theta\,\partial_\theta v = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|-1}(|k|\sin\theta + ik\cos\theta)e^{ik\theta} \\ &= c_1(f)(\sin\theta + i\cos\theta)e^{i\theta} + c_{-1}(f)(\sin\theta - i\cos\theta)e^{-i\theta} + r\tilde{v}_2(r,\theta) \\ &= ic_1(f) - ic_{-1}(f) + r\tilde{v}_2(r,\theta).\end{aligned}$$

Thus, $\partial_2 u$ again has a continuous extension to the origin. This little miracle suggests that something deeper is going on. One could prove inductively that this computation works out similarly for derivatives of all orders (this is not a surprise: $\partial_1 u$ solves the same problem with boundary datum $\partial_1 f$, and we did not use the size of the $c_k(f)$ in the computation), but we present another argument.

Notice that the change of variables Φ can be easily inverted for certain functions, in particular:

$$(x+iy)^{|k|} = r^{|k|}e^{i|k|\theta}, \quad (x-iy)^{|k|} = r^{|k|}e^{-i|k|\theta}.$$

Using these identities and splitting the sum into positive and negative frequencies we can express u directly in cartesian coordinates:

$$u(x,y) = v\circ\Phi^{-1} = c_0(f) + \sum_{k>0} c_k(f)(x+iy)^k + \sum_{k>0} c_{-k}(f)(x-iy)^k.$$

One could now check, by summing the derivatives, that this function is C^∞ as long as $x^2+y^2<1$, but we can also recall complex function theory, set $z := x+iy$ and notice that

$$u(x,y) = c_0(f) + \underbrace{\sum_{k>0} c_k(f) z^k}_{:=\phi(z)} + \underbrace{\sum_{k>0} c_{-k}\overline{z}^k}_{=:\psi(z)}.$$

By standard complex analysis $\phi(z)$ is holomorphic in the unit disk and ψ is antiholomorphic in the unit disk. In particular they both are C^∞.