Exercise 11.1. ♣
Which of the following statements are true?

(a) If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$, then necessarily also $f \in L^2(\mathbb{R}^d)$.

(b) If $\lambda \in \mathbb{C}$ is an eigenvalue[1] of $\mathcal{F}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, then necessarily $\lambda \in \{\pm 1, \pm i\}$.

(c) The function $x \to \frac{1}{1+ix^4}$ is an element of the Schwartz class $\mathcal{S}(\mathbb{R})$.

(d) Let $f \in C^\infty(\mathbb{R})$ be a smooth function with all derivatives bounded on $\mathbb{R}$, i.e. $f^{(j)} \in L^\infty(\mathbb{R})$ for all $j \in \mathbb{N}_0$. Then

$$f\psi \in \mathcal{S}(\mathbb{R}), \quad \forall\, \psi \in \mathcal{S}(\mathbb{R}).$$

Hint: Recall the Leibniz formula for higher-order derivatives of products

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Solution:

(a) True. As the (inverse) Fourier transform is an isometry in L^2 (Plancherel's formula), we have

$$\|f\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\hat{f}\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(\mathbb{R}^d)} < \infty.$$

(b) True. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of the Fourier transform and let $f \in L^2(\mathbb{R}^d)$ be its associated eigenvector, i.e. $\mathcal{F}(f) = \lambda f$. We know that the Fourier transform is an isometry on L^2, with inverse $\mathcal{F}^{-1}$. We also note that $\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x)$ for all $f \in L^2$, $x \in \mathbb{R}^d$. Thus, one has $\mathcal{F}^2(f)(x) = f(-x)$, for all $x \in \mathbb{R}^d$ and hence $\mathcal{F}^4(f) = f$ for all $f \in L^2$. Thus, our eigenvalue λ must satisfy $\lambda^4 = 1$, which implies $\lambda \in \{\pm 1, \pm i\}$.

(c) False. Let $f(x) = 1/(1+ix^4)$. For f to belong to the Schwartz space $\mathcal{S}(\mathbb{R})$, one must have $x \mapsto x^n f^{(m)} \in L^\infty(\mathbb{R})$, for all $n, m \in \mathbb{N}_0$. However, if we choose $m = 0$ and $n > 4$, it is clear that

$$\frac{|x^n|}{|1+ix^4|} = \frac{|x|^n}{(1+x^8)^{1/2}} \notin L^\infty(\mathbb{R}),$$

hence f does not belong to $\mathcal{S}(\mathbb{R})$.

(d) True. Take any differential and polynomial order $n, m \geq 0$ and estimate the supremum norm on $\mathbb{R}$:

$$\begin{aligned}\|x^m \partial^n (f\psi)(x)\|_\infty &= \left\| x^m \sum_{k=0}^{n} \binom{n}{k} \partial^{n-k} f(x)\, \partial^k \psi(x) \right\|_\infty \leq \sum_{k=0}^{n} \binom{n}{k} \left\| x^m \partial^{n-k} f(x)\, \partial^k \psi(x) \right\|_\infty \\ &\leq \sum_{k=0}^{n} \binom{n}{k} \left\| \partial^{n-k} f(x) \right\|_\infty \left\| x^m \partial^k \psi(x) \right\|_\infty < \infty,\end{aligned}$$

where we used that $\left\|\partial^{n-k} f(x)\right\|_\infty < \infty$ by assumption on the function f and $\left\|x^m \partial^k \psi(x)\right\|_\infty < \infty$ since ψ is a Schwartz function.

[1] That is to say: there exists some nonzero function $v \in L^2(\mathbb{R}^d)$ such that $\mathcal{F}v = \lambda v$.

Exercise 11.2.
For $\lambda > 0$, define

$$f(x) = \begin{cases} 1 - \lambda^{-1}|x| & \text{for } |x| < \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that

$$\hat{f}(\xi) = \frac{2}{\lambda}\sqrt{\frac{2}{\pi}} \cdot \frac{\sin^2(\frac{\lambda}{2}\xi)}{\xi^2}.$$

(b) Using the Poisson summation formula in the form[2]

$$\sum_{n\in\mathbb{Z}} \hat{f}\big(\sqrt{2\pi}(\alpha+n)\big) = \sum_{n\in\mathbb{Z}} e^{-i2\pi\alpha n} f\big(\sqrt{2\pi}n\big), \quad \forall\, \alpha \in \mathbb{R},$$

and an appropriate choice of λ in the previous part, show that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}, \quad \forall \alpha \in \mathbb{R}\setminus\mathbb{Z}.$$

Solution:
(a) We compute:

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\Big(\int_0^\lambda (1-\tfrac{x}{\lambda})e^{-i\xi x}\,dx + \int_{-\lambda}^0 (1+\tfrac{x}{\lambda})e^{-i\xi x}\,dx\Big) \\
&= \frac{1}{\sqrt{2\pi}}\Big(\int_0^\lambda (1-\tfrac{x}{\lambda})e^{-i\xi x}\,dx + \int_0^\lambda (1-\tfrac{x}{\lambda})e^{i\xi x}\,dx\Big) \\
&= \sqrt{\frac{2}{\pi}}\int_0^\lambda (1-\tfrac{x}{\lambda})\cos(\xi x)\,dx = \sqrt{\frac{2}{\pi}}\cdot\Big(\Big(1-\frac{x}{\lambda}\Big)\frac{\sin(\xi x)}{\xi}\Big|_{x=0}^{x=\lambda} + \frac{1}{\lambda\xi}\int_0^\lambda \sin(\xi x)\,dx\Big) \\
&= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\lambda\xi^2}\big(1-\cos(\lambda\xi)\big) = \sqrt{\frac{2}{\pi}}\cdot\frac{2}{\lambda}\cdot\frac{\sin^2(\frac{1}{2}\lambda\xi)}{\xi^2},
\end{aligned}$$

where we applied partial integration and used $\cos(\theta) = \cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}) = 1 - 2\sin^2(\frac{\theta}{2})$.

(b) We set $\lambda = \sqrt{2\pi}$. Then

$$f(x) = \begin{cases} 1 - \frac{|x|}{\sqrt{2\pi}} & \text{for } |x| < \sqrt{2\pi}, \\ 0 & \text{otherwise.} \end{cases}, \quad \hat{f}(\xi) = \frac{2}{\pi}\cdot\frac{\sin^2(\sqrt{\frac{\pi}{2}}\xi)}{\xi^2}.$$

Thus, for $\alpha \notin \mathbb{Z}$ we find

$$\sum_{n\in\mathbb{Z}} \hat{f}\big(\sqrt{2\pi}(\alpha+n)\big) = \frac{1}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{\sin^2(\pi\alpha+\pi n)}{(\alpha+n)^2} = \frac{\sin^2(\pi\alpha)}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{1}{(\alpha+n)^2},$$

since $\sin^2(\theta+\pi n) = \sin^2(\theta)$ for all $n\in\mathbb{Z}$. On the other hand, we have

$$\sum_{n\in\mathbb{Z}} e^{-i2\pi\alpha n} f\big(\sqrt{2\pi}n\big) = f(0) = 1,$$

[2] This follows from Exercise 10.4 on the last problem set.

since $f(\sqrt{2\pi}n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Thus, the Poisson summation formula gives

$$\frac{\sin^2(\pi\alpha)}{\pi^2} \sum_{n\in\mathbb{Z}} \frac{1}{(\alpha+n)^2} = 1.$$

Exercise 11.3.

(a) Compute

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(x)}{1+x^2} e^{-i\xi x}\, dx, \quad \text{for } \xi \in \mathbb{R} \setminus \{-1, 1\}.$$

Hint: Extend the integral to a contour integral in the complex plane and apply Cauchy's integral formula.

(b) Deduce that the function $x \to \frac{x\sin(x)}{1+x^2}$ is not in $L^1(\mathbb{R})$.

Solution:

(a) We first use $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$ to write

$$\int_{-R}^{R} \frac{x\sin(x)}{1+x^2} e^{-i\xi x}\, dx = \frac{1}{2i} \int_{-R}^{R} \frac{x}{1+x^2} e^{-i(\xi-1)x}\, dx + \frac{1}{2i} \int_{-R}^{R} \frac{x}{1+x^2} e^{-i(\xi+1)x}\, dx.$$

We thus turn our attention to the integral

$$I_R = \int_{-R}^{R} \frac{x}{1+x^2} e^{-i\alpha x}\, dx = \int_{-R}^{R} \frac{x}{(x+i)(x-i)} e^{-i\alpha x}\, dx$$

for $\alpha \in \mathbb{R} \setminus \{0\}$. This can of course be viewed as a complex contour integral over the real interval $[-R, R] \subset \mathbb{C}$. Note that the integrand is a meromorphic function with simple poles at i and $-i$. Denote by γ_R^+ and γ_R^- the following contours

$$\gamma_R^+ = \{z \in \mathbb{C} \,|\, |z| = R,\ \mathrm{Im}(z) > 0\}, \quad \gamma_R^- = \{z \in \mathbb{C} \,|\, |z| = R,\ \mathrm{Im}(z) < 0\},$$

where the half-circles are traversed counter-clockwise. Furthermore, denote by D_R^+ and D_R^- the following closed contours, again traversed counter-clockwise,

$$D_R^+ = [-R, R] \cup \gamma_R^+, \quad D_R^- = [-R, R] \cup \gamma_R^-.$$

Let $\alpha < 0$. Then we have $|e^{-i\alpha z}| = e^{-|\alpha|\mathrm{Im}(z)}$, so the integrand goes to zero on γ_R^+ as $R \to \infty$. We write the integral above as

$$I_R = \int_{D_R^+} \frac{z}{(z+i)(z-i)} e^{-i\alpha z}\, dz - \int_{\gamma_R^+} \frac{z}{(z+i)(z-i)} e^{-i\alpha z}\, dz.$$

Notice that D_R^+ encircles the pole at $z = i$ when $R > 1$. Thus, Cauchy's integral theorem implies that

$$\int_{D_R^+} \frac{z}{(z+i)(z-i)} e^{-i\alpha z}\, dz = 2\pi i \cdot \frac{i}{2i} e^{\alpha} = \pi i e^{\alpha}, \quad \text{when } R > 1.$$

On the other hand, we have

$$\int_{\gamma_R^+} \frac{z}{(z+i)(z-i)} e^{-i\alpha z}\, dz \to 0 \quad \text{as } R \to \infty.$$

Thus, for $\alpha < 0$ we find

$$\lim_{R\to\infty} I_R = \pi i e^{\alpha}.$$

For $\alpha > 0$ we argue similarly. The integrand now vanishes as $R \to \infty$ for $\operatorname{Im}(z) < 0$, so we write

$$I_R = -\int_{D_R^-} \frac{z}{(z+i)(z-i)} e^{-i\alpha z}\, dz + \int_{\gamma_R^-} \frac{z}{(z+i)(z-i)} e^{-i\alpha z}\, dz,$$

where the change in sign is due to the orientation of the contour, and find by using Cauchy's integral formula:

$$\lim_{R\to\infty} I_R = -\pi i e^{-\alpha}.$$

We can write this concisely as

$$\lim_{R\to\infty} I_R = i\pi \operatorname{sgn}(\alpha) e^{-|\alpha|}.$$

Using this result to compute the original integral, we find

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(x)}{1+x^2} e^{-i\xi x}\, dx = \frac{\pi}{2}\big(\operatorname{sgn}(\xi-1)e^{-|\xi-1|} + \operatorname{sgn}(\xi+1)e^{-|\xi+1|}\big).$$

(b) Denote $f(x) = \frac{x\sin(x)}{1+x^2}$ and notice that f is certainly in $L^2(\mathbb{R})$. The Fourier transform of f can be computed as

$$\hat{f}(\xi) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-i\xi x}\, dx,$$

where the limit converges in L^2, see Remark 3.33 in the lecture notes. Thus, in the previous subquestion we found the Fourier transform of f. Notice that $\hat{f}(\xi)$ has discontinuities at $\xi = 1$ and $\xi = -1$, which cannot be removed by a change of $\hat{f}$ on a zero measure set. But if f were in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then its Fourier transform would have a continuous representative. Thus, f cannot be in $L^1(\mathbb{R})$.

Exercise 11.4.

(a) Let $f \in L^1(\mathbb{R})$ have compact support, i.e. there exists some $R > 0$ such that

$$f = 0 \quad \text{almost everywhere in } \mathbb{R} \setminus [-R, R].$$

Show that $\hat{f}$ is analytic on $\mathbb{R}$. That is, for every $\xi_0 \in \mathbb{R}$ the Taylor series of f around ξ_0 converges to f in a neighborhood of ξ_0.

(b) Let $f \in L^1(\mathbb{R})$ be a continuous function, which is not identically zero. Show that f and its Fourier transform cannot both be compactly supported.

Solution:

(a) Let f have support in $[-R, R]$. Then the Fourier transform of f is

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-ix\xi}\, dx.$$

We first give a short proof of analyticity using complex analysis. Note that the integrand above extends to a holomorphic function in $\xi \in \mathbb{C}$ for each fixed $x \in \mathbb{R}$. That is $\xi \in \mathbb{C} \to f(x)e^{-ix\xi}$ is holomorphic. Moreover, for all $\xi \in \mathbb{C}$, we have $|f(x)e^{-ix\xi}| \leq |f(x)|e^{x\mathrm{Im}(\xi)} \leq e^{R|\mathrm{Im}(\xi)|}|f(x)|$ for $x \in [-R, R]$. Thus, $x \to f(x)e^{-ix\xi}$ is L^1 for each $\xi \in \mathbb{C}$. By a theorem from complex analysis, we thus have

$$\xi \to \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-ix\xi}\, dx \quad \text{holomorphic in } \mathbb{C},$$

so the restriction to $\xi \in \mathbb{R}$ gives a real-analytic function.

We now provide a more direct proof. To this end, we estimate the derivatives of $\hat{f}$. By an application of the dominated convergence theorem, see also Proposition 3.21 and Remark 3.23 in the lecture notes, we have

$$\frac{d^n}{d\xi^n} \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} (-ix)^n f(x) e^{-ix\xi}\, dx,$$

where the integral is absolutely convergent thanks to the compact support of f. In particular, we have $\hat{f} \in C^\infty(\mathbb{R})$ and we can estimate the n-th derivative by

$$\Big|\frac{d^n}{d\xi^n} \hat{f}(\xi)\Big| \leq \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} |x|^n |f(x)|\, dx \leq \frac{1}{\sqrt{2\pi}} R^n \int_{-R}^{R} |f(x)|\, dx \leq \frac{\|f\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}} R^n, \quad \forall\, \xi \in \mathbb{R}.$$

Thus, by Taylor's theorem we have

$$\Big|f(\xi) - \sum_{n=0}^{N} \frac{\hat{f}^{(n)}(\xi_0)}{n!}(\xi - \xi_0)^n\Big| = \Big|\frac{\hat{f}^{(N+1)}(\eta)}{(N+1)!}(\xi - \xi_0)^{N+1}\Big| \leq \frac{\|f\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}} \cdot \frac{R^{N+1}|\xi - \xi_0|^{N+1}}{(N+1)!},$$

which converges to zero as $N \to \infty$ for each fixed $\xi \in \mathbb{R}$. Note that in fact the Taylor series around any $\xi_0 \in \mathbb{R}$ converges absolutely for all $\xi \in \mathbb{R}$:

$$\sum_{n=0}^{\infty} \Big|\frac{\hat{f}^{(n)}(\xi_0)}{n!}(\xi - \xi_0)^n\Big| \leq \frac{\|f\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} R^n |\xi - \xi_0|^n = \frac{\|f\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}} e^{R|\xi - \xi_0|} < \infty, \quad \forall\, \xi_0, \xi \in \mathbb{R}.$$

(b) This follows immediately from the first part of this exercise. If f is compactly supported then $\hat{f}$ is analytic on $\mathbb{R}$ and the only compactly supported analytic function on $\mathbb{R}$ is the zero function.

We also provide a more direct argument. By the first part of the exercise, the Taylor series of $\hat{f}$ converges to $\hat{f}$ on all of $\mathbb{R}$. Now choose ξ_0 outside of the support of $\hat{f}$. Then $\hat{f} = 0$ in a neighborhood of ξ_0, so $\hat{f}^{(n)}(\xi_0) = 0$ for all $n \in \mathbb{N}_0$. Thus, we find

$$\hat{f}(\xi) = \sum_{n=0}^{\infty} \frac{\hat{f}^{(n)}(\xi_0)}{n!}(\xi - \xi_0)^n = 0, \quad \forall\, \xi \in \mathbb{R},$$

which implies $f \equiv 0$, contrary to our assumption.

Exercise 11.5.
Given $\phi \in \mathcal{S}(\mathbb{R})$, we consider the differential equation

$$u'(x) + u(x) = \phi(x), \quad \text{for all } x \in \mathbb{R}.$$

(a) Show that, if there is a solution $u \in \mathcal{S}(\mathbb{R})$, then it is the unique solution within the class of Schwartz functions.

(b) Using the Fourier transform and inverse Fourier transform, show that there is indeed a solution $u \in \mathcal{S}(\mathbb{R})$ to the differential equation.

(c) Solve the differential equation again, this time with classical methods (multiply by e^t etc..).

(d) Check that the two results you found are indeed the same.

Solution:

(a) Suppose that $u, v \in \mathcal{S}(\mathbb{R})$ are both solutions to the differential equation. Then the difference $w := u - v \in \mathcal{S}(\mathbb{R})$ solves the following differential equation:

$$w' + w = (u - v)' + (u - v) = u' + u - (v' - v) = \phi - \phi = 0.$$

This is a first order homogeneous linear differential equation with general solution $w(x) = ce^{-x}$ for any $c \in \mathbb{C}$. Since $w \in \mathcal{S}(\mathbb{R})$, we must in particular have that w is bounded on $\mathbb{R}$, which can only be the case if $c = 0$. Thus, we have $w \equiv 0$, forces $u = v$, i.e. the solution is unique within $\mathcal{S}(\mathbb{R})$.

(b) Since both sides of the equation are Schwartz functions, we may take the Fourier transform. We compute

$$\mathcal{F}\phi(\xi) = \mathcal{F}(u + u')(\xi) = \mathcal{F}u(\xi) + i\xi\mathcal{F}u(\xi) = (1 + i\xi)\mathcal{F}u(\xi),$$

where we used that $\mathcal{F}(u')(\xi) = i\xi\mathcal{F}(u)(\xi)$ for any Schwartz function u. Dividing by $1 + i\xi$ (which is never zero!) yields

$$\mathcal{F}u(\xi) = \frac{1}{1 + i\xi}\mathcal{F}\phi(\xi).$$

Since $\phi \in \mathcal{S}(\mathbb{R})$, we know that (Theorem 3.25) $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R})$. Note that $\xi \in \mathbb{R} \to (1 + i\xi)^{-1}$ is a smooth function with all derivatives bounded on $\mathbb{R}$. Multiplication with such a function leaves the Schwartz class invariant, see exercise 11.1(d). Thus, $(1 + i\xi)^{-1}\mathcal{F}\phi(\xi) \in \mathcal{S}(\mathbb{R})$ and we can take the inverse Fourier transform to get

$$u(x) = \mathcal{F}^{-1}\frac{1}{1 + i\xi}\mathcal{F}\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\mathcal{F}\phi(\xi)}{1 + i\xi}e^{i\xi x}\, d\xi.$$

Note moreover that by Theorem 3.25 $u \in \mathcal{S}(\mathbb{R})$, since it is the inverse Fourier transform of a Schwartz function.

(c) Multiply the differential equation by e^x to get

$$e^x\phi(x) = u(x)e^x + u'(x)e^x = (u(x)e^x)'.$$

Since $e^x\phi(x) \to 0$ as $x \to -\infty$ (ϕ is bounded), we can integrate over the interval $(-\infty, x)$ to get

$$u(x)e^x = \int_{-\infty}^{x} e^t \phi(t)\, dt.$$

Finally divide by e^x, which yields the solution

$$u(x) = \int_{-\infty}^{x} e^{t-x} \phi(t)\, dt.$$

(d) We have found two smooth solutions:

$$u_1(x) = \mathcal{F}^{-1}\frac{1}{1+i\xi}\mathcal{F}\phi \quad \text{and} \quad u_2(x) = \int_{-\infty}^{x} e^{t-x}\phi(t)\, dt.$$

We wish to show that $u_1 \equiv u_2$. Notice that if we introduce the $L^1(\mathbb{R})$ function

$$h(x) := e^{-x}\mathbf{1}_{(0,\infty)}(x),$$

then u_2 can be written as a convolution, namely

$$u_2(x) = \int_{\mathbb{R}} h(x-t)\phi(t)\, dt = (h * \phi)(x).$$

Thus, by the properties of the Fourier transform, we find (notice that both h and ϕ are in $L^1(\mathbb{R}^d)$)

$$\mathcal{F}u_2(\xi) = \sqrt{2\pi}\mathcal{F}h(\xi)\mathcal{F}\phi(\xi) = \frac{1}{1+i\xi}\hat{\phi}(\xi) = \mathcal{F}u_1(\xi), \quad \text{for all } \xi \in \mathbb{R}.$$

Since the Fourier transform is injective on $L^1(\mathbb{R})$, we conclude that $u_1 \equiv u_2$. Here, we used that $\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\xi}$ as can be seen by a direct calculation.