Exercise 12.1. ♣
Which of the following statements are true?

(a) Define $f_n(\xi) = e^{-in\xi}\frac{\sin(\xi)}{\sqrt{\pi}\xi}$. Then $\{f_n \,|\, n \in \mathbb{Z} \text{ even}\}$ is an orthonormal system in $L^2(\mathbb{R})$.
Hint: Think in terms of the Fourier transform.

(b) Let $f \in S(\mathbb{R}^d)$ be a function whose Fourier transform is supported in the ball of radius $\epsilon > 0$, i.e. $\text{supp}(\hat{f}) \subset B_\epsilon$. Then we must have

$$\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, dx \geq \frac{d^2}{4}\epsilon^{-2}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

(c) Let $I \subset \mathbb{R}$ be an open interval and $h \in L^\infty(I)$ with h not identically zero. Then the operator

$$T : L^2(I) \to L^2(I), \quad Tf(x) = h(x)f(x)$$

is compact.

(d) Let V be a finite dimensional inner product space and H a Hilbert space. Then any linear operator $T : V \to H$ is compact.

(e) Let (X, d) be a metric space and let $x \in X$. Assume $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ must possess a sub-subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ with $x_{n_{k_j}} \to x$ as $j \to \infty$. Then $x_n \to x$ as $n \to \infty$.

Solution:

(a) True. Recall that $\xi \to \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(\xi)}{\xi}$ is the Fourier transform of $\mathbf{1}_{[-1,1]}$, the characteristic function of the interval $[-1, 1]$. We have $\|\mathbf{1}_{[-1,1]}\|_{L^2} = \sqrt{2}$. So normalizing we find that $f_0(\xi)$ is the Fourier transform of the norm one element $\frac{1}{\sqrt{2}}\mathbf{1}_{[-1,1]} \in L^2(\mathbb{R}^d)$. Recall that the Fourier transform of a shifted function $x \to f(x-n)$ is just $e^{-in\xi}\hat{f}(\xi)$. Thus, f_n is the Fourier transform of $\frac{1}{\sqrt{2}}[n-1, n+1]$. Since $\{\frac{1}{\sqrt{2}}[n-1, n+1] \,|\, n \in \mathbb{Z} \text{ even}\}$ obviously forms an orthonormal system and the Fourier transform is an isometry, the set in question also forms an orthonormal system.

(b) True. We use the Heisenberg inequality. Since $|\xi| < \epsilon$ on the support of $\hat{f}$, we have

$$\int_{\mathbb{R}^d} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi \leq \epsilon^2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi = \epsilon^2 \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Thus, by the Heisenberg inequality, we have

$$\frac{d}{2}\|f\|^2_{L^2} \leq \|xf\|_{L^2}\|\xi\hat{f}\|_{L^2} \leq \epsilon\|f\|_{L^2}\|xf\|_{L^2}.$$

Dividing by the $\|f\|_{L^2}$ and squaring, we find

$$\|xf\|^2_{L^2} = \int_{\mathbb{R}^d} |x|^2|f(x)|^2 \, dx \geq \frac{d^2}{4}\epsilon^{-2}\|f\|^2_{L^2}.$$

(c) False. Take for instance $h(x) = 1$. Then T is the identity operator on $L^2(I)$, which is not compact, since $L^2(I)$ is an infinite dimensional Hilbert space.

(d) True. Since V is finite dimensional, the range of any operator $T : V \to H$ is finite dimensional, and any finite rank operator is compact.

(e) True. Assume by contradiction that there is a $\delta > 0$ such that $d(x_n, x) > \delta$ for infinitely many $n \in \mathbb{N}$. Then we can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that cannot have any sub-sequences converging to x, since x_{n_k} is always at least a distance δ from x. We have found a contradiction, so x_n must converge to x.

Exercise 12.2.
Consider the Heisenberg inequality on $\mathbb{R}$:

$$\|xf(x)\|_{L^2(\mathbb{R})} \cdot \|\xi \hat{f}(\xi)\|_{L^2(\mathbb{R})} \geq \frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}, \quad \forall\, f \in \mathcal{S}(\mathbb{R}).$$

Show that equality holds if and only if $f(x) = Ce^{-\lambda x^2}$ for some $C \in \mathbb{R}$ and $\lambda > 0$.
Hint: When does equality hold for the Cauchy-Schwarz inequality?

Solution: In the proof of the Heisenberg inequality, we used the Cauchy-Schwarz inequality to deduce

$$\Big|\Big\langle xf, \frac{d}{dx}f\Big\rangle_{L^2(\mathbb{R})}\Big| \leq \Big\|xf\Big\|_{L^2(\mathbb{R})}\Big\|\frac{d}{dx}f\Big\|_{L^2(\mathbb{R})}.$$

For equality to hold in the Heisenberg inequality, we need equality to hold in this application of Cauchy-Schwarz, which is only the case if xf and $\frac{d}{dx}f$ are linearly dependent. Thus, we need

$$\frac{d}{dx}f(x) = axf(x)$$

for some $a \in \mathbb{C}$. The general solution to this ODE is $f(x) = Ce^{\frac{a}{2}x^2}$ for some $C \in \mathbb{R}$. If we denote $\lambda = -\frac{a}{2} \in \mathbb{C}$, then $f(x) = Ce^{-\lambda x^2}$. Notice that we must take $\mathrm{Re}(\lambda) > 0$ for f to be a Schwartz function.

We now check if this function actually gives equality in the Heisenberg inequality. Using $f'(x) = -2\lambda x f(x)$, we compute

$$\begin{aligned}
\|xf(x)\|_{L^2(\mathbb{R})}\|\xi\hat{f}(\xi)\|_{L^2(\mathbb{R})} &= \|xf(x)\|_{L^2(\mathbb{R})}\|f'(x)\|_{L^2(\mathbb{R})} = 2|\lambda|\|xf(x)\|^2_{L^2(\mathbb{R})} = 2|\lambda|\int_{\mathbb{R}} xf(x)\overline{xf(x)}\,dx \\
&= 2|\lambda||C|^2\int_{\mathbb{R}} x^2 e^{-\lambda x^2}e^{-\overline{\lambda}x^2}\,dx = 2|\lambda||C|^2\int_{\mathbb{R}} x^2 e^{-2\mathrm{Re}(\lambda)x^2}\,dx \\
&= -\frac{|\lambda||C|^2}{2\mathrm{Re}(\lambda)}\int_{\mathbb{R}} x\frac{d}{dx}e^{-2\mathrm{Re}(\lambda)x^2}\,dx = \frac{|\lambda|}{2\mathrm{Re}(\lambda)}|C|^2\int_{\mathbb{R}} e^{-2\mathrm{Re}(\lambda)x^2}\,dx \\
&= \frac{|\lambda|}{2\mathrm{Re}(\lambda)}\|f\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

where we used partial integration together with the fact that the boundary terms vanish. Thus, we have $\|xf(x)\|_{L^2(\mathbb{R})}\|\xi\hat{f}(\xi)\|_{L^2(\mathbb{R})} = \frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}$ precisely when $\lambda \in (0, \infty)$.

Exercise 12.3.
Let H be a Hilbert space. Denote the set of compact operators from H to H by $\mathcal{K}(H)$ and the set of bounded operators by $\mathcal{B}(H)$.

(a) Show that $\mathcal{K}(H)$ is a linear subspace of $\mathcal{B}(H)$.

(b) Show that $\mathcal{K}(H)$ is a two-sided ideal in $\mathcal{B}(H)$ with respect to composition, that is for any $T \in \mathcal{K}(H)$ and $S \in \mathcal{B}(H)$, we have $ST \in \mathcal{K}(H)$ and $TS \in \mathcal{K}(H)$.

Solution:

(a) Let $T_1, T_2 \in \mathcal{K}(H)$ and $\lambda \in \mathbb{C}$. We show that $T_1 + \lambda T_2 \in \mathcal{K}(H)$. One could use the continuity of $+ : H \times H \to H$ and $\cdot : \mathbb{C} \times H \to H$ and the fact that continuous functions map compact subsets to compact subsets, to show that $\overline{(T_1 + \lambda T_2)(B_1)}$ is compact. Instead we use the converging subsequences definition of compactness. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Since T_1 is compact, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $T_1 x_{n_k}$ converges to some $y_1 \in H$. Now the sequence $(x_{n_k})_{k\in\mathbb{N}}$ is still a bounded sequence so by compactness of T_2 there is a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ with $T_2 x_{n_{k_j}} \to y_2$ for some $y_2 \in H$. We have then found a subsequence of the original sequence with $(T_1+\lambda T_2)x_{n_{k_j}} = T_1 x_{n_{k_j}} + \lambda T_2 x_{n_{k_j}} \to y_1 + \lambda y_2$. Thus, $T_1 + \lambda T_2$ is a compact operator.

(b) Let $T \in \mathcal{K}(H)$ and $S \in \mathcal{B}(H)$ We first consider the operator TS. Note that $S(B_1) \subset H$ is a bounded set, since S is bounded. Thus, by the compactness of T, we have $\overline{T(S(B_1))}$ compact, showing that TS is a compact operator.

Consider now the operator TS. Since S is bounded, it is continuous, and thus maps compact sets to compact sets. By compactness of T, we have $\overline{T(B_1)}$ compact and hence $S(\overline{T(B_1)})$ compact. This shows that $S(T(B_1)) \subset S(\overline{T(B_1)})$ is a subset of a compact set, and thus its closure $\overline{S(T(B_1))}$ is compact.

Exercise 12.4.
Let $(X, \|\cdot\|)$ be an infinite dimensional normed vector space.

(a) Let $Y \subset X$ be a proper closed linear subspace, i.e. $Y \neq X$. Show that there exists $x \in X$ satisfying $\|x\| = 1$ and $\|x - y\| \geq \frac{1}{2}$ for all $y \in Y$.
Hint: Argue that we can find $x_0 \in X$ with $\alpha := \inf_{y\in Y} \|x_0 - y\| > 0$ and $y_0 \in Y$ with $\alpha \leq \|x_0 - y_0\| \leq 2\alpha$. Then consider $x = \|x_0 - y_0\|^{-1}(x_0 - y_0)$.

(b) Show that the closed unit ball $\overline{B_1} \subset X$ is not compact.
Hint: Use the first part of this exercise to construct a sequence $(x_n)_{n\in\mathbb{N}}$ contained in $\overline{B_1}$ and satisfying $\|x_n - x_m\| \geq \frac{1}{2}$ for all $m \neq n$.

Solution:

(a) Choose any $x_0 \in X \setminus Y$. Then we must have $\alpha := \inf_{y\in Y} \|x_0 - y\| > 0$. Otherwise, there would be a sequence in Y converging to x_0, so x_0 would lie in the closure of Y, and hence in Y, since Y is closed. By definition of infimum, there is some $y_0 \in Y$ satisfying $\|x_0 - y_0\| \leq \inf_{y\in Y} \|x_0 - y\|$. Thus, $\alpha \leq \|x_0 - y_0\| \leq 2\alpha$. Now set

$$x = \frac{1}{\|x_0 - y_0\|}(x_0 - y_0).$$

Then $\|x\| = 1$ and for any $y \in Y$, we have

$$\|x - y\| = \big\|\|x_0 - y_0\|^{-1}(x_0 - y_0) - y\big\| = \frac{1}{\|x_0 - y_0\|}\big\|x_0 - y_0 - \|x_0 - y_0\|y\big\| = \frac{1}{\|x_0 - y_0\|}\|x_0 - \tilde{y}\|,$$

where we defined $\tilde{y} = y_0 + \|x_0 - y_0\|y$. Since Y is a linear subspace, we have $\tilde{y} \in Y$, and hence $\|x_0 - \tilde{y}\| \geq \inf_{y \in Y} \|x_0 - y\| = \alpha$. On the other hand, $\|x_0 - y_0\| \leq 2\alpha$ by construction, so

$$\|x - y\| = \frac{1}{\|x_0 - y_0\|}\|x_0 - \tilde{y}\| \geq \frac{\alpha}{2\alpha} = \frac{1}{2}.$$

(b) We will construct a sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|x_n - x_m\| \geq \frac{1}{2}$ for all $n \neq m$. Such a sequence cannot contain a converging subsequence (since any converging subsequence would be a Cauchy sequence). We have then found a sequence contained in $\overline{B_1}$ with no converging subsequences, so $\overline{B_1}$ cannot be compact.

We construct the sequence inductively. Assume we have found $\{x_1, \ldots, x_N\}$ satisfying $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{1}{2}$ for all $1 \leq n, m \leq N$ with $n \neq m$. Consider the N dimensional subspace $Y = \mathrm{Span}\{x_1, \ldots, x_N\} \subset X$. Then Y is closed since it is finite dimensional. Thus, by the first part of the exercise we can find some $x_{N+1} \in X$ satisfying $\|x_{N+1}\| = 1$ and $\|x_{N+1} - y\| \geq \frac{1}{2}$ for all $y \in Y$. In particular, $\|x_{N+1} - x_n\| \geq \frac{1}{2}$ for all $1 \leq n \leq N$. By induction we obtain a sequence with the desired properties.

Exercise 12.5.
Let $f \in L^2(\mathbb{R}^d)$ be a function whose Fourier transform decays at infinity as a negative power, i.e. for some $\alpha, M > 0$ we have

$$|\hat{f}(\xi)| \leq M|\xi|^{-\alpha} \quad \text{for all } |\xi| \geq 1.$$

The goal of this problem is to show that in fact $f \in C^k(\mathbb{R}^d)$ for all non-negative integers $k < \alpha - d$. (More precisely, f has a representative in $C^k(\mathbb{R}^d)$.)

(a) For each $R > 1$, consider the function

$$f_R(x) := (2\pi)^{-d/2} \int_{B_R} \hat{f}(\xi) e^{i\xi x}\, d\xi,$$

compute $\hat{f}_R$ and show that $f_R \to f$ in $L^2(\mathbb{R}^d)$ as $R \to \infty$.

(b) Show that $f_R \in C^\infty(\mathbb{R}^d)$ for any R, but in general $f_R \notin \mathcal{S}(\mathbb{R}^d)$.

(c) Assume now that $\alpha > d$. Using the decay assumption on $\hat{f}$, show that $(f_R)_{R>0}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^d)$. Conclude that $f \in C(\mathbb{R}^d)$ (up to re-definition on a zero measure set).

(d) Applying the same argument to $\partial_{x_j} f_R$, show inductively that $f \in C^k(\mathbb{R}^d)$ whenever $\alpha > d + k$.

Solution:

(a) Note that f_R is just the inverse Fourier transform of $\mathbf{1}_{B_R}\hat{f} \in L^2(\mathbb{R}^d)$. Using the fact that the Fourier transform is an isometry of L^2, we find

$$\hat{f}_R = \mathcal{F}(\mathcal{F}^{-1}(\hat{f}\ \mathbf{1}_{B_R})) = \hat{f}\ \mathbf{1}_{B_R}.$$

We now compute

$$\|f - f_R\|_{L^2}^2 = \left\|\hat{f} - \hat{f}_R\right\|_{L^2}^2 = \int_{\mathbb{R}^d} \left|\hat{f}(\xi)\right|^2 |1 - \mathbf{1}_{B_R}(\xi)|^2 \, d\xi \longrightarrow 0,$$

as $R \to \infty$. Here we used dominated convergence with

$$\left|\hat{f}(\xi)\right|^2 |1 - \mathbf{1}_{B_R}(\xi)|^2 \leq \left|\hat{f}(\xi)\right|^2 \in L^1(\mathbb{R}^d).$$

(b) Note that by the computation above $\operatorname{supp}(\hat{f}_R) \subset B_R$ so $\hat{f}_R \in L^1 \cap L^2$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ with

$$\psi \equiv 1 \text{ on } B_R.$$

Then

$$f_R = \mathcal{F}^{-1}\left(\hat{f}_R\right) = \mathcal{F}^{-1}\left(\hat{f}_R \cdot \psi\right) = (2\pi)^{d/2} f_R * \mathcal{F}^{-1}(\psi).$$

We have $\mathcal{F}^{-1}(\psi) \in \mathcal{S}(\mathbb{R}^d)$, since $\psi \in \mathcal{S}(\mathbb{R}^d)$, and by the properties of convolution we get that $f_R \in C^\infty(\mathbb{R}^d)$ (the convolution of a smooth L^1 function with an L^1 function is smooth). Alternatively, one could argue iteratively using dominated convergence and the fact that derivatives of the integral defining f_R converge when integrated over B_R.

If we had $f_R \in \mathcal{S}(\mathbb{R}^d)$, then also $\hat{f}_R \in \mathcal{S}(\mathbb{R}^d)$. But $\hat{f}_R$ might not even be continuous. Take for example $f(x) = (2\pi)^{-d/2}\exp(-|x|^2/2)$ with $\hat{f} = f$ and

$$\hat{f}_R = \hat{f}\ \mathbf{1}_{B_R} \notin C^0(\mathbb{R}^d).$$

(c) Assume that $\alpha > d$. We take any $1 < R_1 \leq R_2 < \infty$ and estimate

$$\begin{aligned}
\|f_{R_2} - f_{R_1}\|_{L^\infty} &= (2\pi)^{-d/2} \sup_{x \in \mathbb{R}^d} \left| \int_{B_{R_2}} \hat{f}(\xi) e^{i\xi x} \, d\xi - \int_{B_{R_1}} \hat{f}(\xi) e^{i\xi x} \, d\xi \right| \\
&= (2\pi)^{-d/2} \sup_{x \in \mathbb{R}^d} \left| \int_{B_{R_2} \setminus B_{R_1}} \hat{f}(\xi) e^{i\xi x} \, d\xi \right| \\
&\leq (2\pi)^{-d/2} \int_{B_{R_2} \setminus B_{R_1}} |\hat{f}(\xi)| \, d\xi \\
&\leq (2\pi)^{-d/2} M \int_{\{|\xi| \geq R_1\}} |\xi|^{-\alpha} \, d\xi \longrightarrow 0,
\end{aligned}$$

as $R_1 \to \infty$. Here we used dominated convergence with

$$|\xi|^{-\alpha}\ \mathbf{1}_{\mathbb{R}^d \setminus B_{R_1}}(\xi) \leq |\xi|^{-\alpha}\ \mathbf{1}_{\mathbb{R}^d \setminus B_1}(\xi) \in L^1(\mathbb{R}^d), \text{ since } \alpha > d.$$

This shows that $(f_R)_{R>1}$ is a Cauchy sequence in L^∞ and since L^∞ is complete, we have that f_R converges uniformly as $R \to \infty$. Since $f_R \in C(\mathbb{R}^d)$ for each R, the uniform limit must be continuous. On the other hand, we know that $f_R \to f$ in $L^2(\mathbb{R}^d)$ and hence (up to a subsequence) a.e. on $\mathbb{R}^d$, so (up to redefinition on a zero measure set) f must coincide with the uniform limit of f_R and thus be continuous.

(d) Assume that $\alpha > d + k$. Recall the multiindex notation and let $\beta \in \mathbb{N}_0^n$ with $|\beta| \le k$. We have already seen that f is continuous and f_R is smooth for each $R > 1$. Using either a version of Proposition 3.15 for the inverse Fourier transform, or just dominated convergence together with integrability of $\xi^\beta \hat{f}(\xi)\mathbf{1}_{B_R}$, we obtain

$$\partial^\beta f_R = \partial^\beta \mathcal{F}^{-1}(\hat{f}\mathbf{1}_{B_R}) = \mathcal{F}^{-1}\big((i\xi)^\beta \hat{f}\mathbf{1}_{B_R}\big) = (2\pi)^{-d/2} \int_{B_R} (i\xi)^\beta \hat{f}(\xi) e^{i\xi x}\, d\xi,$$

where we used the facts that $\hat{f}\mathbf{1}_{B_R} \in L^1(\mathbb{R}^d)$ and $\xi^\beta \hat{f}\mathbf{1}_{B_R} \in L^1(\mathbb{R}^d)$. Now take $1 < R_1 \le R_2 < \infty$ and estimate as in the previous subquestion:

$$\begin{aligned}
\|\partial^\beta f_{R_2} - \partial^\beta f_{R_1}\|_{L^\infty} &\le (2\pi)^{-d/2} \sup_{x\in\mathbb{R}^d} \Big| \int_{B_{R_2}\setminus B_{R_1}} (i\xi)^\beta \hat{f}(\xi) e^{i\xi x}\, d\xi \Big| \\
&\le (2\pi)^{-d/2} \int_{B_{R_2}\setminus B_{R_1}} |\hat{f}(\xi)|\, |\xi|^k \, d\xi \\
&\le (2\pi)^{-d/2} M \int_{\{|\xi|\ge R_1\}} |\xi|^{-(\alpha-k)} \, d\xi \longrightarrow 0,
\end{aligned}$$

as $R_1 \to \infty$, where we used dominated convergence with

$$|\xi|^{-(\alpha-k)}\, \mathbf{1}_{\mathbb{R}^d\setminus B_{R_1}}(\xi) \le |\xi|^{-(\alpha-k)}\, \mathbf{1}_{\mathbb{R}^d\setminus B_1}(\xi) \in L^1(\mathbb{R}^d), \text{ since } \alpha - k > d.$$

This shows that $(\partial^\beta f_R)_{R>1}$ forms a Cauchy sequence in L^∞ and hence, by the completeness of L^∞, that $\partial^\beta f_R$ converges uniformly on $\mathbb{R}^d$ as $R \to \infty$ for each $|\beta| \le k$. Recall from Analysis I that since both f_R and its derivatives converge uniformly on $\mathbb{R}^d$, we have $f \in C^k(\mathbb{R}^d)$ and $\partial^\beta f_R$ converges to $\partial^\beta f$ for each $|\beta| \le k$.