Exercise 13.1. ♣
Which of the following statements are true?

(a) If $T : H \to H$ is a compact operator on a Hilbert space, then $0 \in \mathrm{EV}(T)$.

(b) If $T : H \to H$ is a compact operator on a Hilbert space, then $0 \in \sigma(T)$.

(c) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then the operator

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad T\big((x_n)_{n\in\mathbb{N}}\big) = (\alpha_n x_n)_{n\in\mathbb{N}}$$

is compact.

(d) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then the spectrum of the operator

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad T\big((x_n)_{n\in\mathbb{N}}\big) = (\alpha_n x_n)_{n\in\mathbb{N}}$$

satisfies $\sigma(T) = \{\alpha_n\}_{n\in\mathbb{N}} \cup \{0\}$.

(e) Let H be a Hilbert space and let $v_1, \dots v_n, w_1, \dots, w_n \in H$. Then $T(x) = \sum_{k=1}^n \langle x,\, v_k\rangle\, w_k$ defines a bounded linear operator $T : H \to H$.

Solution:

(a) False. See Exercise 13.2 for a counterexample in infinite dimensions.

(b) The way the question is stated this is false. A counterexample is the identity operator on a finite dimensional Hilbert space. If H is infinite dimensional, then the statement is true, see Theorem 4.38 in the lecture notes.

(c) False. Take for instance $\alpha_n = 1$ for all $n \in \mathbb{N}$. One can show that T is compact if and only if $\alpha_n \to 0$ as $n \to \infty$.

(d) False. The set of eigenvalues is given by $\{\alpha_n\}_{n\in\mathbb{N}}$, but any accumulation point of this set is also in the spectrum since $\sigma(T)$ must be closed. So take for instance $\alpha_n = 1 - \frac{1}{n}$. Then $1 \in \sigma(T)$ but $1 \neq \alpha_n$ for all n.

(e) True. Linearity is obvious and boundedness follows from the Cauchy-Schwarz inequality:

$$\|T(x)\| \le \sum_{k=1}^n |\langle x,\, v_k\rangle|\|w_k\| \le \|x\| \sum_{k=1}^n \|v_k\|\|w_k\|.$$

Exercise 13.2.
Consider the map $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$T((x_k)_{k\in\mathbb{N}}) = \left(\frac{x_k}{k}\right)_{k\in\mathbb{N}}.$$

(a) Show that T is a continuous linear operator and determine its norm.

(b) Show that T is the limit (with respect to the operator norm) of a sequence of finite rank operators. Is T compact?

(c) Determine the set of eigenvalues and the spectrum of T.

Solution:

(a) Linearity is immediate. We readily estimate

$$\|T(x)\|_{\ell^2}^2 = \sum_{k=1}^{\infty} \frac{x_k^2}{k^2} \leq \sum_{k=1}^{\infty} x_k^2 = \|x\|_{\ell_2}^2.$$

This shows that T is a bounded, i.e. continuous, operator with operator norm $\|T\| \leq 1$. Defining $e_1 = (1, 0, \dots)$, we have $T(e_1) = e_1$. Thus, in fact $\|T\| = 1$.

(b) For every $m \geq 1$, define $T_m \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ as

$$T_m((x_k)_{k\in\mathbb{N}}) = \left(x_1, \frac{x_2}{2}, \dots, \frac{x_m}{m}, 0, 0, \dots\right).$$

It is clear that the range of T_m is contained in an m-dimensional subspace of $\ell^2(\mathbb{N})$, i.e. T_m is a finite rank operator. We estimate

$$\begin{aligned}
\|T - T_m\| &= \sup_{\|x\|_{\ell^2}=1} \|T(x) - T_m(x)\|_{\ell^2} \\
&= \sup_{\|x\|_{\ell^2}=1} \sqrt{\sum_{k\geq m+1} \frac{x_k^2}{k^2}} \\
&\leq \frac{1}{(m+1)} \sup_{\|x\|_{\ell^2}=1} \sqrt{\sum_{k\geq m+1} x_k^2} \\
&\leq \frac{1}{m+1}.
\end{aligned}$$

Thus, $\|T - T_m\| \to 0$ as $m \to \infty$, i.e. $T_m \to T$ as operators. Since the finite rank operators T_m are compact and the class of compact operators is closed in the topology of the operator norm, T is compact.

(c) An eigenvalue of T is an element $x \in \ell^2(\mathbb{N})$, $x \neq 0$ such that

$$T(x) = \lambda x$$

for some constant $\lambda \in \mathbb{C}$. Passing to the coefficients, the eigenvalue equation reads

$$\frac{x_k}{k} = \lambda x_k, \quad \forall\, k \in \mathbb{N}. \tag{1}$$

Since $x \neq 0$, there is $k \in \mathbb{N}$ with $x_k \neq 0$, which implies $\lambda = 1/k$. But then (1) forces $x_j = 0$ for every $j \neq k$. In particular, the eigenvectors of T are the elements e_k of the standard basis and the corresponding eigenvalues are $1/k$. Lastly, using compactness of T, Theorem 4.38 implies that the spectrum is

$$\sigma(T) = \{0\} \cup \{1/k\}_{k\geq 1}.$$

Exercise 13.3.
Let $\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}^+ = \{q \in \mathbb{Q},\, q > 0\}$, i.e. $n \in \mathbb{N} \to r_n \in \mathbb{Q}^+$ is a bijection. Let H be a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ a Hilbert basis in H. Define the linear operator

$$T : H \to H, \quad T(x) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{r_n} \langle x,\, e_n\rangle\, e_n.$$

(a) Show that T is a bounded operator and compute $\|T\|$.

(b) Determine the set of eigenvalues $\mathrm{EV}(T)$ and the spectrum $\sigma(T)$ of T.

(c) Is T a compact operator?

Solution:

(a) Note that $(\frac{1}{2})^q < 1$ for all $q \in \mathbb{Q}^+$. Thus, for any $x \in H$, we have

$$\|T(x)\|^2 = \sum_{n\in\mathbb{N}} \left(\frac{1}{2}\right)^{2r_n} |\langle x,\, e_n\rangle|^2 \le \sum_{n\in\mathbb{N}} |\langle x,\, e_n\rangle|^2 = \|x\|^2,$$

where we used that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis. On the other hand, for any $\epsilon > 0$ we can choose $q \in \mathbb{Q}^+$ small enough so that $1 - (\frac{1}{2})^q < \epsilon$. Let $n \in \mathbb{N}$ be such that $r_n = q$. Then we have $\|e_n\| = 1$ and $\|T(e_n)\| = (\frac{1}{2})^{r_n} > 1 - \epsilon$. Since ϵ was arbitrary, this shows that $\|T\| = \sup_{\|x\|=1} \|T(x)\| = 1$.

(b) For each $n \in \mathbb{N}$, we have $T(e_n) = (\frac{1}{2})^{r_n} e_n$, as follows from the orthonormality of the basis. Thus, $(\frac{1}{2})^{r_n}$ is an eigenvalue for all n, i.e. $\{(\frac{1}{2})^q \,|\, q \in \mathbb{Q}^+\} \subset \mathrm{EV}(T)$. On the other hand, if λ is an eigenvalue of T with eigenvector x, then $Tx = \lambda x$. Taking the inner product with e_n, we find

$$\langle Tx,\, e_n\rangle = (\tfrac{1}{2})^{r_n} \langle x,\, e_n\rangle = \lambda \langle x,\, e_n\rangle\,.$$

Thus, for each $n \in \mathbb{N}$ either $\lambda = (\frac{1}{2})^{r_n}$ or $\langle x,\, e_n\rangle = 0$, which implies that $(\frac{1}{2})^{r_n}$ are precisely the eigenvalues of T with associated eigenvector e_n. So we have

$$\mathrm{EV}(T) = \left\{(\tfrac{1}{2})^q \,|\, q \in \mathbb{Q}^+\right\}.$$

To find the spectrum, note that $\mathrm{EV}(T) \subset \sigma(T)$. Now notice that $\mathrm{EV}(T)$ is dense in $[0, 1]$ (since $\mathbb{Q}^+$ is dense in $\mathbb{R}^+$). The spectrum $\sigma(T)$ is a closed subset of $\mathbb{C}$, see Proposition 4.36 in the lecture notes, so we must have $[0, 1] \subset \sigma(T)$. On the other hand, if $\lambda \notin [0, 1]$ then there exists $\delta > 0$ so that $|\lambda - (\frac{1}{2})^{r_n}| \ge \delta$ for all $n \in \mathbb{N}$. Using $x = \sum_{n\in\mathbb{N}} \langle x,\, e_n\rangle\, e_n$, we find

$$(T - \lambda \cdot \mathrm{Id})x = \sum_{n\in\mathbb{N}} \left((\tfrac{1}{2})^{r_n} - \lambda\right) \langle x,\, e_n\rangle\, e_n.$$

Define the operator

$$S : H \to H, \quad S(x) = \sum_{n\in\mathbb{N}} \left((\tfrac{1}{2})^{r_n} - \lambda\right)^{-1} \langle x,\, e_n\rangle\, e_n.$$

Since $|(\frac{1}{2})^{r_n} - \lambda|^{-1} \le \delta^{-1}$ for all n, this indeed defines a bounded operator on H and $S(T - \lambda \cdot \mathrm{Id}) = (T - \lambda \cdot \mathrm{Id})S = \mathrm{Id}$. Thus, $T - \lambda \cdot \mathrm{Id}$ is invertible, i.e. $\lambda \notin \sigma(T)$. This shows that

$$\sigma(T) = [0, 1].$$

Remark: The values $\lambda \in \sigma(T) \setminus \mathrm{EV}(T)$ lie in spectrum, because the operator $(T - \lambda \cdot \mathrm{Id})$ is not surjective. It is injective, otherwise λ would be an eigenvalue. In this case, one can actually show that the range $\mathrm{Im}(T - \lambda \cdot \mathrm{Id}) \neq H$ is dense in H.

(c) T is not compact. This follows for instance from Theorem 4.38 in the lecture notes. Indeed, the only accumulation point of the spectrum of a compact operator is 0, whereas $\sigma(T)$ has the entire interval $[0, 1]$ as accumulation points. Also, for a compact operator the spectrum away from 0 coincides with the set of eigenvalues away from 0, which is not the case for T.

Exercise 13.4.

In this exercise you will construct a compact operator, which is in some sense a right inverse for the Laplace operator $-\frac{d^2}{dx^2}$ on $[0, \pi]$.

(a) Let $f \in L^2([0, \pi]; \mathbb{R})$ and consider the following differential equation with boundary conditions:

$$\begin{cases} -u'' = f, \\ u(0) = u(\pi) = 0. \end{cases} \tag{2}$$

Recall from Exercise 7.3 that $B = \{\sqrt{2/\pi}\sin(kx)\}_{k\in\mathbb{N}}$ is a Hilbert basis for $L^2([0, \pi]; \mathbb{R})$. Write both u and f as Fourier series of sines on $[0, \pi]$ and formally find a solution u of (2).

(b) Show that for $f \in L^2([0, \pi]; \mathbb{R})$, the formal solution satisfies $u \in L^2([0, \pi]; \mathbb{R}) \cap C([0, \pi])$ and the boundary condition $u(0) = u(\pi) = 0$ is satisfied.

(c) Prove that the map assigning f to the formal solution u of (2)

$$T\colon L^2([0, \pi]; \mathbb{R}) \to L^2([0, \pi]; \mathbb{R}), \quad T(f) = u$$

is a continuous linear operator, which is moreover self-adjoint and compact.

(d) Show that the set of functions

$$\{u \in C^2([0, \pi]; \mathbb{R}),\, u(0) = u(\pi) = 0\}$$

is contained in the image of T. What is a sufficient condition on the Fourier coefficients of f to make sure that $u = T(f)$ is C^2, i.e. that it is a classical solution of (2)?

(e) We have shown that the map

$$P : \mathrm{Im}(T) \to L^2([0, \pi]; \mathbb{R}), \quad P(u) = f, \quad \text{where } u = T(f),$$

is an extension of the operator $-\frac{d^2}{dx^2}$ from the subspace

$$\{u \in C^2([0, \pi]; \mathbb{R}),\, u(0) = u(\pi) = 0\} \subset L^2([0, \pi]; \mathbb{R})$$

to the larger subspace $\mathrm{Im}(T) \subset L^2([0, \pi]; \mathbb{R})$. By the Spectral Theorem for compact operators, we know that there is a Hilbert basis consisting of eigenfunctions of T. Show that the same eigenfunctions are eigenfunctions of P.

Remark: We have found a Hilbert basis of eigenfunctions for the Laplace operator $-\frac{d^2}{dx^2}$ with Dirichlet boundary conditions, i.e. an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}} \subset L^2([0, \pi]; \mathbb{R})$ and eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\begin{cases} -\varphi_n'' = \lambda_n e_n, \\ \varphi_n(0) = \varphi_n(\pi) = 0. \end{cases}$$

Of course we've been working with said eigenfunctions since the beginning of this exercise, what are they? A similar strategy can be used to find a Hilbert basis of eigenfunctions for the Laplace operator with Dirichlet boundary conditions on more complicated domains.

Solution:

(a) Since f is a real-valued L^2 function on $[0, \pi]$, it has a Fourier series expansion in terms of sines, see Exercise 7.3:

$$f(x) = \sum_{k=1}^{\infty} a_k(f) \sin(kx), \quad \text{where} \quad a_k(f) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx.$$

We make a similar ansatz for u:

$$u(x) = \sum_{k=1}^{\infty} a_k(u) \sin(kx), \quad a_k(u) = \frac{2}{\pi} \int_0^{\pi} u(x) \sin(kx)\, dx.$$

Note that we choose a Fourier series in terms of sines in order to satisfy the boundary conditions at $0, \pi$. Formally plugging this into equation (2), we find

$$-u''(x) = \sum_{k=1}^{\infty} k^2 a_k(u) \sin(kx) = \sum_{k=1}^{\infty} a_k(f) \sin(kx) = f(x).$$

Thus, for every $f \in L^2([0, \pi], \mathbb{R})$ the function defined by the expression

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k(f)}{k^2} \sin(kx) \tag{3}$$

is a formal solution to (2).

(b) Note that for $f \in L^2([0, \pi]; \mathbb{R})$ the series in (3) converges in L^2. Indeed, this can be seen from square-summability of the Fourier coefficients; one can for instance argue with Theorem 2.13 and the fact that

$$\sum_{k=1}^{\infty} \left\| \frac{a_k(f)}{k^2} \sin(kx) \right\|^2_{L^2([0,\pi])} = \frac{\pi}{2} \sum_{k=1}^{\infty} \left(\frac{a_k(f)}{k^2} \right)^2 \leq \frac{\pi}{2} \sum_{k \geq 1} a_k(f)^2 < \infty,$$

where we used that $\| \sin(kx) \|^2_{L^2([0,\pi])} = \frac{\pi}{2}$ for all $k \in \mathbb{N}$. Furthermore, the absolute summability of the Fourier coefficients of u implies that u is continuous. Indeed, the partial sums in (3) are smooth and converge uniformly to u on $[0, \pi]$, since

$$\sum_{k=1}^{\infty} \left\| \frac{a_k(f)}{k^2} \sin(kx) \right\|_{L^\infty([0,\pi])} = \sum_{k=1}^{\infty} \frac{|a_k(f)|}{k^2} \leq \Big(\sum_{k=1}^{\infty} |a_k(f)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} \frac{1}{k^4} \Big)^{\frac{1}{2}} < \infty.$$

Thus, $u \in C([0, \pi])$ and pointwise convergence of the Fourier series gives $u(0) = u(\pi) = 0$ (since $\sin(0) = \sin(k\pi) = 0$ for all $k \in \mathbb{N}$).

(c) Linearity of the map $T : f \to u$ follows immediately from linearity of the Fourier coefficients $a_k(f)$. For continuity of the operator we use orthonormality of $\sqrt{2/\pi}\sin(kx)$ to estimate as above:

$$\|u\|^2_{L^2([0,\pi])} = \left\|\sum_{k=1}^{\infty} \frac{a_k(f)}{k^2}\sin(kx)\right\|^2_{L^2([0,\pi])} = \frac{\pi}{2}\sum_{k=1}^{\infty}\left(\frac{a_k(f)}{k^2}\right)^2 \le \frac{\pi}{2}\sum_{k\ge 1} a_k(f)^2 = \|f\|_{L^2([0,\pi])}.$$

Thus, $\|T\| \le 1$. Self-adjointness follows directly, since for $f, g \in L^2([0,\pi],\mathbb{R})$ we have

$$\langle Tf,\, g\rangle = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{a_k(f)}{k^2}a_k(g) = \frac{\pi}{2}\sum_{k=1}^{\infty} a_k(f)\frac{a_k(g)}{k^2} = \langle f,\, Tg\rangle\,,$$

where we once again used the orthonormality of $\{\sqrt{2/\pi}\sin(kx)\}_{k\in\mathbb{N}}$. Compactness follows as in Exercise 13.2. Define

$$T_n : L^2([0,\pi];\mathbb{R}) \to L^2([0,\pi];\mathbb{R}), \quad T_n(f)(x) = \sum_{k=1}^{n}\frac{a_k(f)}{k^2}\sin(kx).$$

Then the range of T_n is contained in $\mathrm{Span}\{\sin(x), \sin(2x), \ldots, \sin(nx)\}$, i.e. T_n is a finite rank operator for each $n \in \mathbb{N}$, and

$$\begin{aligned}\|T - T_n\| &= \sup_{\|f\|_{L^2}=1}\|T(f) - T_n(f)\|_{L^2([0,\pi])} = \sup_{\|f\|_{L^2}=1}\Big(\frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{a_k(f)^2}{k^4}\Big)^{\frac{1}{2}} \\ &\le \frac{1}{(n+1)^2}\sup_{\|f\|_{L^2}=1}\Big(\frac{\pi}{2}\sum_{k=n+1}^{\infty}a_k(f)^2\Big)^{\frac{1}{2}} \le \frac{1}{(n+1)^2} \to 0\end{aligned}$$

as $n \to \infty$. Thus, T is compact as the limit of a sequence of finite rank operators.

(d) Assume that $u \in C^2([0,\pi];\mathbb{R})$ and $u(0) = u(\pi) = 0$. Then in particular $u'' \in C([0,\pi];\mathbb{R}) \subset L^2([0,\pi];\mathbb{R})$, so u'' has a Fourier expansion in terms of sines. The coefficients satisfy

$$\begin{aligned}a_k(u'') &= \frac{2}{\pi}\int_0^{\pi} u''(x)\sin(kx)\,dx = \frac{2}{\pi}\Big(u'(x)\sin(kx)\big|_{x=0}^{x=\pi} - k\int_0^{\pi}u'(x)\cos(kx)\,dx\Big) \\ &= \frac{2}{\pi}\Big(k\cdot u(x)\cos(kx)\big|_{x=0}^{x=\pi} - k^2\int_0^{\pi}u(x)\sin(kx)\,dx\Big) \\ &= -k^2 a_k(u),\end{aligned}$$

where in the first line we used that $\sin(0) = \sin(k\pi) = 0$ and in the second line we used that $u(0) = u(\pi) = 0$. This is similar to the strategy of Theorem 2.22 in the lecture notes. However, note that the boundary conditions are important here since $\sin(kx)$ is not actually periodic on $[0,\pi]$ when k is odd. Now $a_k(u'')$ is square summable, so

$$\sum_{k=1}^{\infty}k^2|a_k(u)| = \sum_{k=1}^{\infty}|a_k(u'')| < \infty.$$

Thus, we can define an L^2 functions $f(x) = -u''(x) = \sum_{k=1}^{\infty}k^2 a_k(u)\sin(kx) \in L^2([0,\pi];\mathbb{R})$ and we see from the definition of the operator T that $T(f) = u$.

For the second part, note that by Theorem 2.26 in the lecture notes (or a slight variation thereof), if the Fourier coefficients of $u = T(f)$ satisfy

$$\sum_{k=1}^{\infty} k^2 |a_k(u)| = \sum_{k=1}^{\infty} k^2 \frac{|a_k(f)|}{k^2} = \sum_{k\geq 1} |a_k(f)| < \infty$$

then u is of class C^2 and $-u'' = f$. Thus, absolute summability of the Fourier coefficients of f is enough to ensure that u is a classical solution to (2).

(e) Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of eigenfunctions for T with associated eigenvalues $\{\mu_n\}_{n\in\mathbb{N}}$. Notice that $\mu_n \neq 0$ for all $n \in \mathbb{N}$. Indeed, T has trivial kernel, since if

$$T(f) = \sum_{k=1}^{\infty} \frac{a_k(f)}{k^2} \sin(kx) = 0,$$

then we must have $a_k(f) = 0$ for all k, so $f = 0$. Hence, $\varphi_n = \frac{1}{\mu_n} T\varphi_n$ for all n, so φ_n lies in the image of T. Thus, the operator P is well-defined on φ_n and we have

$$P\varphi_n = \frac{1}{\mu_n} PT\varphi_n = \frac{1}{\mu_n}\varphi_n.$$

This shows that the φ_n form an orthonormal basis of eigenfunctions of P with associated eigenvalues $\lambda_n = \frac{1}{\mu_n}$. Of course the eigenfunctions are given explicitly by $\sqrt{2/\pi}\,\sin(nx)$ with eigenvalue n^2.