ANALYSIS IV - EXAM #1 - 90 MIN

Problem 1. Let H be a *complex* vector space and consider a *function* $\langle\cdot,\cdot\rangle\colon H\times H\to\mathbb{C}$.

(a) Define what it means that the pair $(H,\langle\cdot,\cdot\rangle)$ is a complex Hilbert space.

From now on assume that $(H,\langle\cdot,\cdot\rangle)$ is indeed an Hilbert space.

(b) Let $V\subset H$ be a vector subspace. Providing all the necessary assumptions, state the projection theorem on V. More precisely, define the closest-point projection operator $\pi_V\colon H\to V$ and characterize $\pi_V(x)$ (the projection of a point x) by a suitable orthogonality condition. No proofs are required.

(c) Assume $H:=L^2(\mathbb{R})$ with the standard L^2 scalar product and $V:=\{\text{odd functions in } L^2(\mathbb{R})\}$. After checking the necessary assumptions, prove that

$$\pi_V(f)(x)=\frac{f(x)-f(-x)}{2}.$$

Problem 2.

(a) Compute the Fourier transform of $f(t):=\mathcal{X}_{[-1/2,1/2]}(t)$, $t\in\mathbb{R}$.

(b) Given u,v in $L^1(\mathbb{R})$, express $\mathcal{F}(u*v)$ in terms of $\hat{u}$ and $\hat{v}$. Prove rigorously your formula and specify whether $\mathcal{F}(u*v)$ is computed in the L^1 or in the L^2 sense.

(c) Check that $g(t):=(f*f)(t)=(1-|t|)_+$ for all $t\in\mathbb{R}$ and compute $\hat{g}$.

(d) Does $\hat{g}$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$?

Problem 3. Consider the heat-type PDE

$$\text{(P)}\qquad \partial_t u=\cos(t)\partial_{xx}u,\ \text{in } (0,T)\times\mathbb{R},\quad u(0^+,x)=f(x)\text{ for all } x\in\mathbb{R},$$

where

- $T>0$ is a given "final time",
- $u(t,x)$ is assumed to be real-valued and 2π-periodic in the x variable, that is $u(t,x)=u(t,x+2\pi)$ for all $t\in(0,T)$ and $x\in\mathbb{R}$,
- $f(x)$ is a given initial condition which is also 2π-periodic.

Complete the following tasks:

(a) Assuming you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ construct a formal solution w of (P) as a Fourier series in the x variable with t-dependent coefficients.

(b) Check that, if $\int_{-\pi}^{\pi}|f|^2<\infty$ and $T<\pi$, then $w\colon(0,T)\times\mathbb{R}\to\mathbb{R}$ is indeed a well-defined continuous function.

(c) Show that also the initial condition is met in the sense that

$$\lim_{t\downarrow 0}\|w(t,\cdot)-f\|_{L^2(-\pi,\pi)}=0.$$

(d) Show that w is in fact of class C^2 (in both variables) and solves the equation

$$\partial_t w=\cos(t)\partial_{xx}w\ \text{in } (0,T)\times\mathbb{R}.$$

Please turn the page!

You can give for granted the following facts:

- The definition of vector space over $\mathbb{C}$.
- A function $f\colon \mathbb{R} \to \mathbb{C}$ is *odd* if $f(-x) = -f(x)$.
- If $E \subset \mathbb{R}$, then $\mathcal{X}_E\colon \mathbb{R} \to \{0,1\}$ is the indicator function of E.
- If $w(t)$ is a real-valued function then $w(t)_+$ denotes the *positive part* of w, that is $w(t)_+ := \max\{w(t), 0\}$.
- The Fourier transform in $\mathbb{R}^d$ (under suitable assumptions) is given by
$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, dx.$$
- The convolution of $f, g\colon \mathbb{R} \to \mathbb{C}$ (under suitable assumptions) is given by
$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, dy.$$
- For a 2π periodic function $f\colon \mathbb{R} \to \mathbb{C}$ the kth fourier coefficient is given by
$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \qquad \text{for each } k \in \mathbb{Z}.$$
Under suitable assumption f can be expressed as
$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{ikx}.$$