ANALYSIS IV - EXAM #2 - 90 MIN

Problem 1. Let H be a complex vector space endowed with a *function* $\langle \cdot, \cdot \rangle \colon H \times H \to \mathbb{R}$.

(a) Define what it means that $(H, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space.

(b) Assuming $(H, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space, define what is an "orthonormal system" of H.

(c) Assuming $(H, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space and let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal system. Prove that the following are equivalent

(i) $\operatorname{span}\{e_k\}_{k\in\mathbb{N}}$ is dense in H;

(ii) for all $x \in H$, it holds

$$x = \sum_{k=0}^{\infty} \langle x, e_k \rangle e_k;$$

(iii) $x = 0$ is the only vector $x \in H$ such that

$$\langle x, e_k \rangle = 0 \quad \text{for all } k \in \mathbb{N}.$$

Problem 2.

(a) State (but don't prove) Plancherel's identity for functions in the Schwarz class $\mathcal{S}(\mathbb{R})$.

(b) Define the Fourier transform of an $L^2(\mathbb{R})$ function and show that the definition is well-posed.

(c) Compute the Fourier transform of $\mathbb{R} \ni x \mapsto e^{-|x|}$.

(d) Compute the value of

$$\int_{\mathbb{R}} (1+t^2)^{-2}\, dt.$$

Problem 3. Consider the wave-type PDE

$$\partial_{tt} u - \partial_{xx} u = 2\cos(2x) \text{ in } \mathbb{R} \times \mathbb{R}$$

where $u(t,x)$ is assumed to be 2π periodic in the x variable and satisfy the initial conditions

$$u(0,\cdot) = 0, \quad u'(0,\cdot) = f.$$

(a) Assuming you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ construct a formal solution w of as a Fourier series in the x variable with t-dependent coefficients.

(b) Check that, if $\sum_{k\in\mathbb{Z}} k|c_k(f)| < +\infty$ then $w \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is well-defined, of class C^2 and satisfies

$$\partial_{tt} w - \partial_{xx} w = 2\cos(2x).$$

(c) Under the same assumptions on $\{c_k(f)\}$, show that $\lim_{t\to 0} \|w(t,\cdot)\|_{L^\infty(-\pi,\pi)} = 0$ and $\lim_{t\to 0} \|\partial_t w(t,\cdot) - f\|_{L^\infty(-\pi,\pi)} = 0$.