Question 4

[10 Points]

Q4 (i) [3 Points] Prove that the norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ does not arise from an inner product in the space $L^\infty(\mathbb{R}^n)$.

Q4 (ii) [3 Points] State and prove the Riesz Representation Theorem.

Q4 (iii) [4 Points] Consider the functional $T_{\alpha,\beta}\colon L^2(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ defined by

$$T_{\alpha,\beta}(g) \coloneqq \int_{\mathbb{R}^n} (g(x) + \beta)(1 + |x|)^{-\alpha} \, \mathrm{d}x.$$

Determine for which pairs $(\alpha, \beta) \in (0, +\infty) \times \mathbb{R}$ the functional $T_{\alpha,\beta}$ is linear and for which pairs it is continuous. For those pairs for which $T_{\alpha,\beta}$ is both linear and continuous, determine its Riesz representation.

Question 5

[8 Points]

Q5 (i) [3 Points] Given two functions $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, express $\mathcal{F}(\varphi * \psi)$ in terms of $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$ and prove the statement.

Q5 (ii) [2 Points] Let $\Phi : \mathbb{R} \to \mathbb{R}$ be the Gaussian distribution, i.e. $\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Compute $\Psi \coloneqq \Phi * \Phi$.

Q5 (iii) [3 Points] Compute the Fourier transform of $h(x) \coloneqq x\Phi(x) = \frac{1}{\sqrt{2\pi}} x e^{-x^2/2}$.

Reminder: recall that the Fourier transform in $\mathbb{R}$ is defined, for suitable functions $f : \mathbb{R} \to \mathbb{C}$, as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, \mathrm{d}x, \qquad \xi \in \mathbb{R}.$$

Question 6

[12 Points]

Consider the Schrödinger-type PDE

$$\begin{cases} iu_t + u + u_{xx} = 0 & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = f(x) & \text{in } \mathbb{R}, \end{cases} \tag{P}$$

where u is assumed to be a real-valued 2π-periodic function on $\mathbb{R}$ and f is also 2π-periodic.

Q6 (i) [3 Points] Assuming that you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ of f, construct a formal solution w to (P) as a Fourier series in the x variable with t dependent coefficients.

Q6 (ii) [4 Points] Check that if $f \in C^\infty_{per}([-\pi,\pi])$, then the function w constructed is well defined, of class C^∞ and solves

$$iw_t + w + w_{xx} = 0 \quad \forall (t,x) \in (0,+\infty) \times \mathbb{R}$$

Q6 (iii) [3 Points] Show that the initial condition is met, in the sense that

$$\lim_{t\to 0^+} \|w(t,\cdot) - f\|_{L^\infty} = 0.$$

Q6 (iv) [2 Points] Does the limit

$$\lim_{t\to\infty} w(t,\cdot)$$

always exist? Is it finite?