Question 4

[10 Points]

Q4 (i) [3 Points] Prove that the norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ does not arise from an inner product in the space $L^\infty(\mathbb{R}^n)$.

Solution:

The parallelogram identity

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 2\|f\|_\infty^2 + 2\|g\|_\infty^2$$

does not hold for all $f, g \in L^\infty$. As a counterexample, consider for instance two disjoint open sets $A, B \subset \mathbb{R}^n$ and take $f = \chi_A$, $g = \chi_B$. Then

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 2 < 4 = 2\|f\|_\infty^2 + 2\|g\|_\infty^2.$$

Q4 (ii) [3 Points] State and prove the Riesz Representation Theorem.

Solution:

Theorem (Riesz) Let H be a Hilbert space and $T\colon H \to \mathbb{K}$ be a continuous linear functional on H. Then, there exists $x_0 \in H$ such that $T(x) = \langle x, x_0 \rangle$.

Proof: We can suppose that $\ker T \neq H$, otherwise the representation is given by $x_0 = 0$. $\ker T$ is a closed subspace by continuity and linearity. By the non-triviality of the orthogonal space there exists $z_0 \in H$ such that $\langle z, z_0 \rangle = 0$ for every $z \in \ker T$ and $T(z_0) = 1$. For every $x \in H$, $x - T(x)z_0 \in \ker T$ since

$$T(x - T(x)z_0) = T(x) - T(x) = 0.$$

Thus,

$$0 = \langle x - T(x)z_0, z_0 \rangle = \langle x, z_0 \rangle - T(x)\|z_0\|^2$$

and the proof is concluded choosing $x_0 = z_0/\|z_0\|^2$.

Q4 (iii) [4 Points] Consider the functional $T_{\alpha,\beta}\colon L^2(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}$ defined by

$$T_{\alpha,\beta}(g) \coloneqq \int_{\mathbb{R}^n} (g(x)+\beta)(1+|x|)^{-\alpha}\,\mathrm{d}x.$$

Determine for which pairs $(\alpha,\beta) \in (0,+\infty)\times\mathbb{R}$ the functional $T_{\alpha,\beta}$ is linear and for which pairs it is continuous. For those pairs for which $T_{\alpha,\beta}$ is both linear and continuous, determine its Riesz representation.

Solution:

If $\beta \neq 0$ the functional is not linear since $T_{\alpha,\beta}(0) \neq 0$. By Cauchy–Schwarz inequality

$$\begin{aligned} |T_{\alpha,0}(g)| &= \left| \int_{\mathbb{R}^n} \frac{g(x)}{(1+|x|)^{\alpha}} dx \right| \\ &\leq \|g\|_{L^2(\mathbb{R}^n)} \left(\int_0^{+\infty} \frac{\omega_n r^{n-1}}{(1+r)^{2\alpha}} dr \right)^{1/2} \end{aligned}$$

and the last integral is finite if $\alpha > n/2$, so in this case when $\beta = 0$ we obtain a linear and bounded functional. For $\alpha \leq n/2$, if the functional were linear bounded, the function $(1+|x|)^{-\alpha}$ would have to be its Riesz representation, and this is only in L^2 for $\alpha > n/2$. Hence $T_{\alpha,\beta}$ is a continuous linear functional if and only if $\beta = 0$ and $\alpha > n/2$, and in those cases, its Riesz representation is given by taking the scalar product with $(1+|x|)^{-\alpha}$.

Note: if $\beta \neq 0$, the necessary and sufficient condition for the functional to be continuous (and even well-defined) is that $\alpha > n$: indeed, only then is the term $\beta(1+|x|)^{-\alpha}$ summable.

Question 5

[8 Points]

Q5 (i) [3 Points] Given two functions $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, express $\mathcal{F}(\varphi * \psi)$ in terms of $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$ and prove the statement.

Solution:

It holds

$$\mathcal{F}(\varphi * \psi) = \sqrt{2\pi}\mathcal{F}(\varphi)\mathcal{F}(\psi).$$

Indeed, we have

$$\begin{aligned}
\mathcal{F}(\varphi * \psi)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)g(y)dy \right) e^{-ix\xi} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)g(y)e^{-i(x-y)\xi}e^{-iy\xi} dy \right) dx \\
&= \int_{\mathbb{R}} \hat{f}(\xi)g(y)e^{-iy\xi} dy \\
&= \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi),
\end{aligned}$$

where the second-to-last inequality holds because of Fubini's theorem, which is applicable since

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| f(x-y)g(y)e^{-i(x-y)\xi}e^{-iy\xi} \right| dy \right) dx \le \|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$$

by Young's inequality .

Q5 (ii) [2 Points] Let $\Phi : \mathbb{R} \to \mathbb{R}$ be the Gaussian distribution, i.e. $\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Compute $\Psi := \Phi * \Phi$.

Solution:

By the previous formula, knowing that $\hat{\Phi} = \Phi$, we have

$$\hat{\Psi}(\xi) = \sqrt{2\pi}\hat{\Phi}^2(\xi) = \sqrt{2\pi}\Phi^2(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2} = \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{2}\xi)^2/2}.$$

Knowing that the inverse Fourier transform of $\hat{f}(\lambda \cdot)$ is $\frac{1}{\lambda} f\left(\frac{\cdot}{\lambda}\right)$, we get that

$$\Psi(x) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-(x/\sqrt{2})^2/2} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-x^2/4}.$$

Q5 (iii) [3 Points] Compute the Fourier transform of $h(x) \coloneqq x\Phi(x) = \frac{1}{\sqrt{2\pi}} x e^{-x^2/2}$.

Solution:

It holds

$$\partial_\xi \hat{f}(\xi) = \mathcal{F}(-ixf(x)),$$

thus,

$$\hat{h}(\xi) = \mathcal{F}(x\Phi(x))(\xi) = i\partial_\xi \hat{\Phi}(\xi) = i\frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}(-\xi) = \frac{-i\xi}{\sqrt{2\pi}} e^{-\xi^2/2}.$$

Reminder: recall that the Fourier transform in $\mathbb{R}$ is defined, for suitable functions $f : \mathbb{R} \to \mathbb{C}$, as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, \mathrm{d}x, \qquad \xi \in \mathbb{R}.$$

Question 6

[12 Points]

Consider the Schrödinger-type PDE

$$\begin{cases} iu_t + u + u_{xx} = 0 & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = f(x) & \text{in } \mathbb{R}, \end{cases} \tag{P}$$

where u is assumed to be a real-valued 2π-periodic function on $\mathbb{R}$ and f is also 2π-periodic.

Q6 (i) [3 Points] Assuming that you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ of f, construct a formal solution w to (P) as a Fourier series in the x variable with t dependent coefficients.

Solution:

Write $w(t,x) = \sum_{k\in\mathbb{Z}} e^{ikx} w_k(t)$ and examine the equation satisfied by w:

$$\sum_{k\in\mathbb{Z}} e^{ikx} \left(iw_k'(t) + w_k - k^2 w_k \right) = 0.$$

Thus each term is zero and we get the ODEs $w_k'(t) = i(1-k^2)w_k$ whose solution is $w_k(t) = e^{i(1-k^2)t} w_k(0)$. Since $u(0,x) = f(x)$, we get the initial conditions $w_k(0) = c_k(f)$, hence the formal solution is

$$w(t,x) = \sum_{k\in\mathbb{Z}} e^{ikx} e^{i(1-k^2)t} c_k(f).$$

Q6 (ii) [4 Points] Check that if $f \in C^\infty_{per}([-\pi,\pi])$, then the function w constructed is well defined, of class C^∞ and solves

$$iw_t + w + w_{xx} = 0 \quad \forall (t,x) \in (0,+\infty) \times \mathbb{R}$$

Solution:

Since $f \in C^\infty_{per}$, its Fourier coefficients decay faster then any polynomial, i.e. $\sup_{k\in\mathbb{Z}} |k|^\alpha |c_k(f)| < \infty$ for every $\alpha \in \mathbb{N}$. We will show that

$$\sum_{k\in\mathbb{Z}} \sup_{(0,\infty)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t) e^{ikx})| < \infty$$

for every pair of integers $p, q \geq 0$. This will prove that the series converges uniformly, along with all its derivatives, on compact subsets of $(0,+\infty) \times \mathbb{R}$ and therefore that it is a classical solution, since differentiation can be performed term by term. Observe that

$$\begin{aligned} \sup_{(0,\infty)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t) e^{ikx})| &= |c_k(f)(ik)^q (i(1-k^2))^p e^{i((1-k^2)t+kx)}| \\ &\leq |c_k(f)||k^q(1-k^2)^p|. \end{aligned}$$

Set $\ell = 2p + q + 2$ and observe that $|c_k(f)| \leq C_\ell (1+|k|)^{-\ell}$ for every $k \in \mathbb{Z}$. Thus we get

$$\sum_{k\in\mathbb{Z}} \sup_{(0,\infty)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t) e^{ikx})| \leq \sum_{k\in\mathbb{Z}} |c_k(f)||k^q(1-k^2)^p| \leq C_\ell \sum_{k\in\mathbb{Z}} (1+|k|)^{-2} < \infty,$$

which means that the above sum converges uniformly to the continuous function $\partial_t^p \partial_x^q w$.

Q6 (iii) [3 Points] Show that the initial condition is met, in the sense that

$$\lim_{t\to 0^+} \|w(t,\cdot) - f\|_{L^\infty} = 0.$$

Solution:

We check that

$$\begin{aligned}\sup_{x\in\mathbb{R}} |w(t,x) - f(x)| &= \sup_{x\in\mathbb{R}} \left| \sum_{k\in\mathbb{Z}} w_k(t) e^{ikx} - \sum_{k\in\mathbb{Z}} c_k(f) e^{ikx} \right| \\ &\leq \sum_{k\in\mathbb{Z}} \left| c_k(f) e^{i(1-k^2)t} - c_k(f) \right| \\ &= \sum_{k\in\mathbb{Z}} |c_k(f)| |e^{i(1-k^2)t} - 1|.\end{aligned}$$

Taking the limit as $t \to 0^+$ yields the conclusion once we show that the passage of the limit in the sum in the right-hand side is justified. This fact follows from the Dominated Convergence Theorem, since $|e^{i(1-k^2)t} - 1| \leq 2$ and

$$\sum_{k\in\mathbb{Z}} |c_k(f)|\, |e^{i(1-k^2)t} - 1| \leq 2 \sum_{k\in\mathbb{Z}} |c_k(f)| \leq 2C_2 \sum_{k\in\mathbb{Z}} (1+|k|)^{-2} < \infty.$$

Q6 (iv) [2 Points] Does the limit

$$\lim_{t\to\infty} w(t,\cdot)$$

always exist? Is it finite?

Solution:

The limit $\lim_{t\to\infty} w(t,\cdot)$ does not exist in general, in any sense. As an example, let $f = 1$ and observe that the corresponding solution is

$$w(t,x) = e^{it},$$

which does not converge as $t \to +\infty$.