Question 4

[8 Points]

Q4 (i) [2 Points] Give the definition of separable complex Hilbert space.

Q4 (ii) [3 Points] Show that all separable, complex, infinite-dimensional Hilbert spaces are isometric to each other.

Q4 (iii) [3 Points] Let $H = L^2(\mathbb{R})$ and let

$$V \coloneqq \{\varphi \in H : \varphi(x) = -\varphi(-x)\}.$$

Taking for granted that V is closed, show that $V^\perp$ is closed and prove that

$$\pi_{V^\perp}(\varphi)(x) = \tfrac{1}{2}(\varphi(x) + \varphi(-x)) \qquad \forall \varphi \in H.$$

Question 5

[11 Points]

Q5 (i) [4 Points] For a function $f \in L^1(\mathbb{R}^n)$, define its Fourier transform $\hat{f}$ and show that it is well defined at every point of $\mathbb{R}^n$. Show also that $\hat{f}$ is continuous.

Q5 (ii) [4 Points] Compute the Fourier transform of $g \in L^2(\mathbb{R})$ given by $g(x) = e^{-|x|}$.

Q5 (iii) [3 Points] Compute the Fourier transform of $f(x) \coloneqq x g(x)$.

Question 6

[11 Points]

Consider the heat-type PDE

$$\begin{cases} u_t = u_{xx} + 2\sin(2x), & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}, \end{cases} \tag{P}$$

where $u(t,\cdot)$ is assumed to be 2π-periodic for each t and f is also 2π-periodic.

Q6 (i) [**3 Points**] Assuming that you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ of f, construct a formal solution w to (P) as a Fourier series in the x variable with t-dependent coefficients.

Q6 (ii) [**4 Points**] Check that if $f \in L^2([-\pi,\pi])$ the function w constructed is well-defined, of class C^2 and solves

$$w_t = w_{xx} + 2\sin(2x) \quad \forall (t,x) \in (0,+\infty) \times \mathbb{R}$$

in the classical sense.

Q6 (iii) [**4 Points**] Show that the initial condition is met, in the sense that

$$\lim_{t\to 0^+} \|w(t,\cdot) - f\|_{L^2} = 0.$$