Question 4

[8 Points]

Q4 (i) [2 Points] Give the definition of separable complex Hilbert space.

Solution:

See the script.

Q4 (ii) [3 Points] Show that all separable, complex, infinite-dimensional Hilbert spaces are isometric to each other.

Solution:

Let H be a separable Hilbert space, and let $(u_k)_{k\in\mathbb{N}}$ be a Hilbert basis of H. By transitivity, it is enough to see that H is isometric to ℓ^2, so fix $(e_k)_{k\in\mathbb{N}}$ the standard basis of ℓ^2, and define the map $T : \ell^2 \to H$ as follows: given a finite sum $\sum' a_k e_k$, we let $T\left(\sum' a_k e_k\right) = \sum' a_k u_k$. The map thus defined is a linear isometry from a dense subspace onto its image, and thus extends to a linear isometry $T : \ell^2 \to H$ onto its image.

It only remains to check that T is surjective: given $u \in H$, consider $a_k = \langle u, u_k\rangle$ and observe that $T\left(\sum_{k\in\mathbb{N}} a_k e_k\right) = \sum_{k\in\mathbb{N}}\langle u, u_k\rangle u_k = u$. $\square$

Q4 (iii) [3 Points] Let $H = L^2(\mathbb{R})$ and let

$$V \coloneqq \{\varphi \in H : \varphi(x) = -\varphi(-x)\}.$$

Taking for granted that V is closed, show that $V^\perp$ is closed and prove that

$$\pi_{V^\perp}(\varphi)(x) = \tfrac{1}{2}(\varphi(x) + \varphi(-x)) \qquad \forall \varphi \in H.$$

Solution:

To see that $V^\perp$ is closed, let $(\varphi_k)_k$ be a sequence in $V^\perp$ which converges in L^2 to $\varphi \in H$; given $v \in V$, we have that $\langle \varphi, v\rangle = \lim_{n\to\infty}\langle \varphi_k, v\rangle = 0$, as $\langle \cdot, v\rangle$ is continuous on H, so $\varphi \in V^\perp$ too.

Given any $\varphi \in H$, we first show that $x \mapsto \frac{1}{2}(\varphi(x) + \varphi(-x))$ is in $V^\perp$: given any $v \in V$,

$$\begin{aligned}\left\langle \frac{1}{2}(\varphi(\cdot) + \varphi(-\cdot)), v \right\rangle &= \frac{1}{2}\int \varphi(x)v(x)\,dx + \frac{1}{2}\int \varphi(-x)v(x)\,dx \\ &= \frac{1}{2}\int \varphi(x)v(x)\,dx - \frac{1}{2}\int \varphi(-x)v(-x)\,dx \\ &= \frac{1}{2}\int \varphi(x)v(x)\,dx - \frac{1}{2}\int \varphi(x)v(x)\,dx = 0,\end{aligned}$$

where we have used the change of variable $x \rightsquigarrow -x$. To characterize the orthogonal projection, we also need to show that $\varphi - \pi_{V^\perp}\varphi \in V$:

$$\varphi(x) - \pi_{V^\perp}\varphi(x) = \varphi(x) - \frac{1}{2}(\varphi(x) + \varphi(-x)) = \frac{1}{2}\varphi(x) - \frac{1}{2}\varphi(-x)$$

and

$$\varphi(-x) - \pi_{V^\perp}\varphi(-x) = \varphi(-x) - \frac{1}{2}(\varphi(-x) + \varphi(x)) = \frac{1}{2}\varphi(-x) - \frac{1}{2}\varphi(x) = -\left(\varphi(x) - \pi_{V^\perp}\varphi(x)\right),$$

so indeed $\varphi(\cdot) - \pi_{V^\perp}\varphi(\cdot) \in V$ and $\pi_{V^\perp}$ is the orthogonal projection. $\square$

Question 5

[11 Points]

Q5 (i) [4 Points] For a function $f \in L^1(\mathbb{R}^n)$, define its Fourier transform $\hat{f}$ and show that it is well defined at every point of $\mathbb{R}^n$. Show also that $\hat{f}$ is continuous.

Solution:

The Fourier transform is defined as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x}\, dx.$$

Since $f \in L^1$ we have that for every $\xi \in \mathbb{R}^n$, also $f(x)e^{-i\xi\cdot x}$ is in L^1 and therefore $\hat{f}(\xi)$ is well defined. Moreover, if we have a sequence $\xi_k \to \xi$, then since the functions $f(x)e^{-i\xi_k\cdot x}$ are uniformly bounded by $|f(x)|$, which is summable, and converge pointwise to $f(x)e^{-i\xi\cdot x}$, by the dominated convergence theorem we can pass the limit inside the integral and get that $\hat{f}(\xi_k) \to \hat{f}(\xi)$, which implies the continuity of $\hat{f}$. □

Q5 (ii) [4 Points] Compute the Fourier transform of $g \in L^2(\mathbb{R})$ given by $g(x) = e^{-|x|}$.

Solution:

We compute

$$\begin{aligned}
\sqrt{2\pi}\hat{g}(\xi) &= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\xi x}\, dx = \int_{-\infty}^{0} e^{x} e^{-i\xi x}\, dx + \int_{0}^{\infty} e^{-x} e^{-i\xi x}\, dx \\
&= \int_{0}^{\infty} e^{-x(1-i\xi)}\, dx + \int_{0}^{\infty} e^{-x(1+i\xi)}\, dx \\
&= \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2},
\end{aligned}$$

hence $\hat{g}(\xi) = \sqrt{2/\pi}(1+\xi^2)^{-1}$. □

Q5 (iii) [3 Points] Compute the Fourier transform of $f(x) := xg(x)$.

Solution:

Recall the formula relating the deritvative of the Fourier transform of g with the Fourier transform of $xg(x)$: $\hat{g}'(\xi) = -i\hat{f}(\xi)$. Hence

$$\hat{f}(\xi) = ig'(\xi) = -\sqrt{\frac{2}{\pi}} \frac{2i\xi}{(1+\xi^2)^2}.$$ □

Question 6

[11 Points]

Consider the heat-type PDE

$$\begin{cases} u_t = u_{xx} + 2\sin(2x), & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}, \end{cases} \tag{P}$$

where $u(t,\cdot)$ is assumed to be 2π-periodic for each t and f is also 2π-periodic.

Q6 (i) [3 Points] Assuming that you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ of f, construct a formal solution w to (P) as a Fourier series in the x variable with t-dependent coefficients.

Solution:

Write $w(t,x) = \sum_{k\in\mathbb{Z}} a_k(t)e^{ikx}$ formally and substitute into the PDE. In order to do that, first recall that

$$\sin(2x) = \frac{e^{2ix} - e^{-2ix}}{2i}$$

and also that $w_t(t,x) = \sum_{k\in\mathbb{Z}} a_k'(t)e^{ikx}$ and $w_{xx}(t,x) = -\sum_{k\in\mathbb{Z}} k^2 a_k(t)e^{ikx}$. Thus, since the (formal) Fourier decomposition is unique we may equate the two sides and get the following set of ODEs:

$$a_k'(t) = -k^2 a_k(t) \qquad \text{for } k \neq \pm 2$$

$$a_2'(t) = -4a_2(t) - i, \qquad a_{-2}'(t) = -4a_{-2}(t) + i$$

with initial conditions $a_k(0) = c_k(f)$ for all $k \in \mathbb{Z}$. The ODEs for $k \neq \pm 2$ are immediate to solve and we get $a_k(t) = c_k(f)e^{-k^2 t}$. For $k = \pm 2$ the homogeneous solution is the same, and the particular solution is just a constant. Hence we get

$$a_2(t) = k_2 e^{-4t} - \frac{i}{4} \Longrightarrow k_2 = \frac{i}{4} + c_2(f) \Longrightarrow a_2(t) = \left(\frac{i}{4} + c_2(f)\right)e^{-4t} - \frac{i}{4}$$

and

$$a_{-2}(t) = k_{-2}e^{-4t} + \frac{i}{4} \Longrightarrow k_{-2} = -\frac{i}{4} + c_{-2}(f) \Longrightarrow a_{-2}(t) = \left(-\frac{i}{4} + c_{-2}(f)\right)e^{-4t} + \frac{i}{4}.$$

Putting all together, the formal solution is

$$\begin{aligned} w(t,x) &= \sum_{k\in\mathbb{Z}} a_k(t)e^{ikx} = \sum_{k\in\mathbb{Z}} c_k(f)e^{-kt^2}e^{ikx} + \frac{i}{4}(e^{-4t} - 1)e^{2ix} - \frac{i}{4}(e^{-4t} - 1)e^{-2ix} \\ &= \sum_{k\in\mathbb{Z}} c_k(f)e^{-kt^2}e^{ikx} + \frac{1}{2}(1 - e^{-4t})\sin(2x). \end{aligned}$$

□

Q6 (ii) [4 Points] Check that if $f \in L^2([-\pi,\pi])$ the function w constructed is well-defined, of class C^2 and solves

$$w_t = w_{xx} + 2\sin(2x) \quad \forall (t,x) \in (0,+\infty) \times \mathbb{R}$$

in the classical sense.

Solution:

Since the finite Fourier sums are smooth, it is enough to check that the Fourier series of $\partial^\alpha w$ converge locally uniformly to $\partial^\alpha w$ for each multi-index α with $|\alpha| \le 2$. In turn, it suffices to check that for every $\delta > 0$, the Fourier sums of these functions converge absolutely in the C^0 norm on $[\delta,\infty) \times \mathbb{R}$, that is, for $j + \ell \le 2$,

$$\sum_{k\in\mathbb{Z}} \left\| \partial_t^j \partial_x^\ell (a_k(t) e^{ikx}) \right\|_{C^0([\delta,\infty)\times\mathbb{R})} < \infty.$$

Indeed,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \left\| \partial_t^j \partial_x^\ell (a_k(t) e^{ikx}) \right\|_{C^0([\delta,\infty)\times\mathbb{R})} &\le C + \sum_{k\in\mathbb{Z}} \left\| \partial_t^j \partial_x^\ell (c_k(f) e^{-k^2 t} e^{ikx}) \right\|_{C^0([\delta,\infty)\times\mathbb{R})} \\
&= C + \sum_{k\in\mathbb{Z}} \left\| (-k^2)^j (ik)^\ell c_k(f) e^{-k^2 t} e^{ikx} \right\|_{C^0([\delta,\infty)\times\mathbb{R})} \\
&\le C + \sum_{k\in\mathbb{Z}} |k|^{2j+\ell} e^{-k^2\delta} |c_k(f)| \\
&\le C + \left(\sum_{k\in\mathbb{Z}} |k|^{2j+\ell} e^{-k^2\delta} \right)^{1/2} \left(\sum_{k\in\mathbb{Z}} |c_k(f)|^2 \right)^{1/2} < +\infty,
\end{aligned}$$

where we have used Cauchy–Schwarz, Parseval's identity (together with the fact that $f \in L^2([-\pi,\pi])$), and the convergence of the series for each j,ℓ. Thanks to the C^2 convergence of the Fourier series of w, we can exchange the sums and the partial derivatives and deduce that w is not only a formal solution, but also a classical one. □

Q6 (iii) [4 Points] Show that the initial condition is met, in the sense that

$$\lim_{t\to 0^+} \|w(t,\cdot) - f\|_{L^2} = 0.$$

Solution:

By using Parseval's identity, we see that our problem is equivalent to showing that

$$\|w(t,\cdot) - f\|^2_{L^2([-\pi,\pi])} = 2\pi \sum_{k\in\mathbb{Z}} |a_k(t) - c_k(f)|^2 = 2\pi \sum_{k\in\mathbb{Z}} |a_k(t) - c_k(f)|^2 \xrightarrow{t\to 0} 0.$$

It is enough to show it for a sequence $t_j \to 0$, and we will do that using the Dominated Convergence Theorem for sums on $\mathbb{Z} \setminus \{\pm 2\}$. Clearly $a_k(t) \to c_k(f)$ as $t \to 0$ for every $k \in \mathbb{Z}$. Moreover, $|a_k(t)| = |c_k(f)| e^{-k^2 t} \le |c_k(f)|$ for each $k \ne 2$. Hence the sequences $(|a_k(t) - c_k(f)|^2)_{k\in\mathbb{Z}\setminus\{\pm 2\}}$ are bounded by the sequence $4|c_k(f)|^2$, which is summable, and we can pass to the limit. □