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Algebraic Topology II

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Problem set 4

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Show that for a connected non-orientable manifold M there is a unique orientable double cover of M.

3. Show that for any connected closed orientable n-manifold M there is a degree 1 map $f : M \to S^n$.

4. Let $f : M \to N$ be a map between connected closed orientable manifolds and suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is a disjoint union of open balls $B_1, \ldots, B_k \subset M$ which each get mapped homeomorphically onto B. Show that the degree of f is $\sum \varepsilon_i$, where ε_i is ± 1 according to whether $f|_{B_i} : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

5. Let M, N be closed connected orientable manifolds and let $f : M \to N$ a p-sheeted covering map. Show that f has degree $\pm p$.

6. Consider a pair of spaces $(X, Y) = (Q \cup R, S \cup T)$ such that $S \subset Q$, $T \subset R$ and such that the interiors of Q, R cover X and the interiors of S, T cover Y. Show that there is a relative Mayer-Vietoris LES

$$\cdots \to H_n(Q \cap R, S \cap T) \to H_n(Q, S) \oplus H_n(R, T) \to H_n(X, Y) \to H_{n-1}(Q \cap R, S \cap T) \to \cdots$$

Hint: Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(S \cap T) \arrow[r] \arrow[d] & S_n(S) \oplus S_n(T) \arrow[r] \arrow[d] & S_n(S + T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(Q \cap R) \arrow[r] \arrow[d] & S_n(Q) \oplus S_n(R) \arrow[r] \arrow[d] & S_n(Q + R) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(Q \cap R, S \cap T) \arrow[r] \arrow[d] & S_n(Q, S) \oplus S_n(R, T) \arrow[r] \arrow[d] & S_n(Q + R, S + T) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the horizontal maps are of the form $x \mapsto (x, -x)$ resp. $(x, y) \mapsto x + y$; $S_n(Q+R)$ is the subgroup of $S_n(X)$ consisting of sums of chains in Q and R (and similarly for $S_n(S+T)$), and $S_n(Q + R, S + T)$ denotes the quotient of $S_n(Q + R)$ by $S_n(S + T)$. Show first that the third row is a chain complex. Then show it is exact by considering the diagram as a short exact sequence of chain complexes. Finally deduce the existence of the LES.

7. The goal of this exercise is to prove the following theorem:

Theorem. *Let M be a connected non-compact manifold of dimension n. Then $H_i(M;R)=0$ for all $i\geq n$.*

Let $i \geq n$ and $a=[z]\in H_i(M;R)$. We will omit R from the notation. Let $U\subset M$ be an open neighbourhood of image(z) such that $\overline{U}$ is compact. Set $V = M\backslash\overline{U}$.

 (a) Recall the LES for triples and apply it to $(M, U\cup V, V)$.

 (b) Use (a) to show $H_i(U)\cong H_i(U\cup V, V)=0$ for $i>n$. Deduce that $a=0$ in case $i>n$. *Hint:* Use the second part of the first lemma from lecture 12B, to see that some groups in the LES vanish.

 (c) To prove the theorem for $i=n$, consider the section $x\mapsto s(x)=L_{M,x}(a)$ of $\widetilde{M}_R\to M$. Show that $s=0$. *Hint:* Note that it's enough to show $s(x_0)=0$ for some $x_0\in M$.

 (d) Deduce that $[z]=0$ in $H_n(M|\overline{U})=H_n(M,V)$. *Hint:* Apply the first part of the first lemma from lecture 12B.

 (e) Apply the LES in (a) to see that $[z]=0\in H_n(U)$ and deduce that $a=0$.

8. Given two disjoint connected n-manifolds M_1 and M_2, their connected sum $M_1\#M_2$ can be constructed by deleting the interiors of closed n-balls $B_1\subset M_1$ and $B_2\subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. Show that for closed connected orientable n-manifolds M_1, M_2 there are isomorphisms

$$H_i(M_1)\oplus H_i(M_2)\cong H_i(M_1\#M_2)$$

for $0<i<n$.

9. Show that if a closed orientable manifold of dimension $2n$ has $H_{n-1}(M)$ torsion-free then $H_n(M)$ is also torsion-free.

10. Compute the cup product structure of $H^*((S^2\times S^8)\#(S^4\times S^6))$, and in particular show that the only non-trivial cup products are those forced by Poincaré duality.

11. Show that if M is a compact connected non-orientable 3-manifold, $H_1(M)$ is infinite.

12. Prove that every map $f:\mathbb{C}P^n\to\mathbb{C}P^n$ has $\deg f=k^n$ for some $k\in\mathbb{Z}$.

13. Let $\alpha\in H^n(S^n)$ be a generator, and define $u=\alpha\times 1, v=1\times\alpha\in H^n(S^n\times S^n)$. Let now $f:S^n\times S^n\to S^n\times S^n$ be a map with $\deg f=\pm1$. Writing $f^*(u)=au+bv$, $f^*v=cu+dv$ and assuming that n is even, prove that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}0 & \pm1\\ \pm1 & 0\end{pmatrix}.$$

14. Let M be a closed connected orientable n-manifold and suppose that there exists a map $f:S^n\to M$ with $\deg f\neq 0$. Prove that $H_*(M;\mathbb{Q})\cong H_*(S^n;\mathbb{Q})$. If $\deg f=\pm1$, prove that $H_*(M;\mathbb{Z})\cong H_*(S^n;\mathbb{Z})$.

15. Prove that if a closed connected orientable manifold M can be written as the union $M=U\cup V$ of two acyclic subsets, then $H_*(M)\cong H_*(S^n)$.