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Algebraic Topology II

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Problem set on tensor products

Let R be a commutative ring. Below we omit R from the notation $\otimes_R$ whenever the ring R is obvious from the context. When no ring R is mentioned, $\otimes$ means tensor product over $\mathbb{Z}$.

1. Let M be an R-module. Show that for every exact sequence of R-modules $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ the sequence
$$M \otimes U \xrightarrow{\mathrm{id}\otimes f} M \otimes V \xrightarrow{\mathrm{id}\otimes g} M \otimes W \to 0$$
is exact. Find a counterexample to the above statement in the case we do not assume that the map g is surjective.
Hint: To prove exactness at $M \otimes V$, construct a left-inverse for an appropriate map $M \otimes V/\mathrm{im}(\mathrm{id} \otimes f) \to M \otimes W$.

2. Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence of R-modules. Assume the sequence splits. Prove that for every R-module M the sequence
$$0 \to U \otimes M \xrightarrow{f\otimes \mathrm{id}} V \otimes M \xrightarrow{g\otimes \mathrm{id}} W \otimes M \to 0$$
is exact.

3. Let M be a free R-module. Show that for every short exact sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ the sequence
$$0 \to M \otimes U \xrightarrow{\mathrm{id}\otimes f} M \otimes V \xrightarrow{\mathrm{id}\otimes g} M \otimes W \to 0$$
is exact.

4. Find counterexamples to the statements in problems 2, resp. 3, in case we drop the assumption that the sequence $0 \to U \to V \to W \to 0$ splits, resp. the module M is free.

5. Let M be an R-module and $J \subset R$ an ideal. Consider the submodule of M generated by $\{a \cdot m | a \in J, m \in M\}$ and denote this submodule by JM. Consider also R/J as an R-module. Prove that
$$(R/J) \otimes_R M \cong M/JM.$$

6. Prove that $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_d$, where $d = gcd(n, m)$.

7. An abelian group G is called *divisible* if for all $g \in G, 0 \neq n \in \mathbb{Z}$, there exists $h \in G$ such that $nh = g$. For example, G is a field of characteristic 0, endowed with the $+$ operation, or $G = \mathbb{R}/\mathbb{Z}$, or $G = \mathbb{Q}/\mathbb{Z}$.

   An abelian group T is calles *torsion* if for all $t \in T$, there exists $0 \neq n \in \mathbb{Z}$ such that $nt = 0$. Prove that if G is divisible and T is torsion then $G \otimes T = 0$.

   This can be generalized to R-modules: An R-module M is called *divisible* if for any $m \in M$ and any $r \in R$, which is not a 0-divisor, there exists $n \in M$ such that $rn = m$. An R-module T is called *torsion*, if for each $m \in M$, there exists $r \in R$ which is not a zero-divisor, such that $rm = 0$. If M is divisible and T is torsion, then $M \otimes_R T = 0$.

8. Prove that there exists isomorphisms

$$\hom_R(U, \hom_R(V, W)) \cong \mathrm{Bilin}_R(U, V; W) \cong \hom_R(U \otimes_R V, W)$$

   for all R-modules U, V, W that are natural in U, in V and in W. Here $\hom_R$ denotes the space of R-module morphism, and $\mathrm{Bil}_R(U, V; W)$ the space of bilinear M-module morphisms $U \times V \to W$.

9. Let U, V be free R-modules of finite rank. Consider $U^* := \hom_R(U, R)$ viewed as an R-module. Prove that there exists an isomorphism

$$U^* \otimes_R V \cong \hom_R(U, V)$$

   which is natural in U and in V.