Solutions to problem set on tensor products

1. Every element of $M \otimes W$ of the form $m \otimes w$ is in the image of $\mathrm{id} \otimes g$ because g is surjective; since every element of $M \otimes W$ is a sum of elements of this form, it follows that $\mathrm{id} \otimes g$ is surjective. By a similar argument one sees that $\mathrm{im}(\mathrm{id} \otimes f) \subseteq \ker(\mathrm{id} \otimes \mathrm{g})$.

   To prove $\ker(\mathrm{id} \otimes \mathrm{g}) \subseteq \mathrm{im}(\mathrm{id} \otimes f) =: I$, consider the map $\phi : M \otimes V/I \to M \otimes W$ induced by $\mathrm{id} \otimes g$, which is well-defined because $I \subseteq \ker(\mathrm{id} \otimes \mathrm{g})$. We now define a map $\psi : M \otimes W \to M \otimes V/I$ which is a left inverse for ϕ, i.e. such that $\psi \circ \phi = \mathrm{id}$; this implies injectivity of ϕ and hence that $\ker(\mathrm{id} \otimes g) \subseteq I$. To define ψ, consider first the map $M \times W \to M \otimes V/I$ defined as follows: It takes (m, w) to $[m \otimes v]$, where $v \in V$ is any element such that $g(v) = w$. This is well-defined and bilinear and hence descends to a map $\psi : M \otimes W \to M \otimes V/I$. We clearly have $\psi \circ \phi = \mathrm{id}$: That's obvious on elements of the form $[m \otimes v]$, and these generate.

   We show that given an exact sequence of R-modules $U \to V \to W$ the tensored sequence is not exact in general. Consider
   $$0 \to \mathbb{Z} \to \mathbb{Z}$$
   where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2 (which is of course not surjective) and consider $M = \mathbb{Z}_2$. Then the induced sequence
   $$0 \to \mathbb{Z}_2 \otimes \mathbb{Z} \cong \mathbb{Z}_2 \to \mathbb{Z}_2 \otimes \mathbb{Z} \cong \mathbb{Z}_2$$
   is not exact anymore, as multiplication by 2 on $\mathbb{Z}_2$ is the 0-map. Here we used the property $U \otimes_R R \cong U$ for any R-module U.

2. In view of problem 1, it is enough to prove injectivity of $f \otimes \mathrm{id}$. Let $j : V \to U$ be a left-inverse to f: $j \circ f = \mathrm{id}$. Then
   $$(j \otimes \mathrm{id}) \circ (f \otimes \mathrm{id}) = (j \circ f) \otimes \mathrm{id} = \mathrm{id},$$
   and hence $j \otimes \mathrm{id}$ is a left-invere to $f \otimes \mathrm{id}$. In particular, $f \otimes \mathrm{id}$ is injective.

3. In view of problem 1, what is left to prove is the injectivity of $\mathrm{id} \otimes f$. Freeness of M means that it has a linearly independent generating set $\{m_i\}_{i \in I}$. Note that every element of $M \otimes U$ can be written as a sum $\sum_{i \in I} m_i \otimes u_i$ and that there is a well-defined map $M \otimes U \to \bigoplus_{i \in I} U$ taking such an element to $(u_i)_{i \in I}$. It follows that $(\mathrm{id} \otimes f)(\sum m_i \otimes u_i) = \sum m_i \otimes f(u_i) = 0$ implies $f(u_i) = 0$ for all i, hence $u_i = 0$ for all i by injectivity of f, and hence $\sum m_i \otimes u_i = 0$.

4. Consider the short exact sequence of $\mathbb{Z}$-modules
   $$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0.$$
   Tensoring with the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ yields the sequence
   $$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \to 0.$$
   This is not exact at the left copy of $\mathbb{Z}/2\mathbb{Z}$ and so this is a counter-example to both 2 and 3. Note that the intial sequence does not split and $\mathbb{Z}/2\mathbb{Z}$ is not a free $\mathbb{Z}$-module.

5. See Lang, *Algebra.*, Chapter XVI, §2, Proposition 2.7.

6. Apply problem 5 to $R = \mathbb{Z}$, $J = n\mathbb{Z}$ and $M = \mathbb{Z}_m$. We get

$$\begin{aligned}\mathbb{Z}_n \otimes \mathbb{Z}_m &\cong \mathbb{Z}_m / \left(n\mathbb{Z} \cdot \mathbb{Z}_m\right) \cong (\mathbb{Z}/m\mathbb{Z}) \,/\, ((n\mathbb{Z} + m\mathbb{Z})/m\mathbb{Z}) \\ &\cong \mathbb{Z}/(n\mathbb{Z} + m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z} \cong \mathbb{Z}_d.\end{aligned}$$

   Alternatively, one can use the universal property of the tensor product.

7. We show that $m \otimes t \in M \otimes T$ vanishes for any $m \in M$ and $t \in T$. Since t is torision, there exists $r \in R$, which is not a zero-divisor and such that $rt = 0$. Since m is divisible by r, there exists $n \in M$ such that $m = rn$. We compute

$$m \otimes t = (rn) \otimes t = r(n \otimes t) = n \otimes (rt) = n \otimes 0 = 0.$$

   Hence $M \otimes T = 0$.

8. See Lang, *Algebra.*, Chapter XVI, Beginning of §2.

9. See Lang, *Algebra.*, Chapter XVI, §5, Corollary 5.5.