Lecture #1A

Algebraic Topology 2

OVERVIEW

* homology with coeffs.: $H_*(X;G)$.
* cohomology: $H^*(X;G)$.
* What's the relation between $H_*(X;G)$ and $H_*(X)\otimes G$?
* Algebraic operations on H_* & H^*, products etc.
* Manifolds, Poincaré duality: relation between $H_*(X)$ & $H^*(X)$.

Tensor products.

Good refs:
* Atiyah–McDonald.
* Lang – "Algebra".

Def. Let R be a commut. ring with unity. Let $\mathcal{U}, V$ be R-modules. A tensor product of $\mathcal{U}$ & V (over R) is an R-module T together with a bilinear map (over R) $\tau : \mathcal{U} \times V \longrightarrow T$ s.t. : $\forall$ R-module W & $\forall$ bilinear map $f : \mathcal{U} \times V \longrightarrow W$ $\exists$ a unique homomorphism $\bar{f} : T \longrightarrow W$ s.t. $\bar{f} \circ \tau = f$

$$\begin{tikzcd} & T \arrow[d, dashed, "\bar{f}\ \exists!"] \\ \mathcal{U} \times V \arrow[ur, "\tau"] \arrow[r, "f"'] & W \end{tikzcd}$$

Lemma. If T exists then it is unique up to iso. in the sense that if $\tau' : \mathcal{U} \times V \longrightarrow T'$ is also a tensor prod. of $\mathcal{U}$ & V then $\exists!$ an iso. $h : T' \longrightarrow T$ s.t. $\tau = h \circ \tau'$

Proof. Exc.

$$\begin{tikzcd} & T' \arrow[d, dashed, "h"', "\cong\ \exists!"] \\ \mathcal{U} \times V \arrow[ur, "\tau'"] \arrow[r, "\tau"'] & T \end{tikzcd}$$

Thm. $\forall$ R-mod. $\mathcal{U}$ & V, a tensor prod of $\mathcal{U}$ & V exists.

Proof. Let M be the free R-module generated by all the pairs (u,v) with $u \in \mathcal{U}$, $v \in V$. Let $N \subset M$ be the submodule generated by the following elements:

$$\left\{ \begin{array}{l} (u+u',v) - (u,v) - (u',v) \\ (u,v+v') - (u,v) - (u,v') \\ (au,v) - a(u,v) \\ (u,av) - a(u,v) \end{array} \right\} \quad \begin{array}{l} u,u' \in \mathcal{U} \\ v,v' \in V \\ a \in R \end{array} \qquad \text{Put } T := M/N.$$

We have an injection of sets $i : \mathcal{U} \times V \longrightarrow M$.

Define: $\tau = \left(\mathcal{U} \times V \xrightarrow{i} M \longrightarrow M/N \right)$.

Exc. check that τ is a bilinear map.

Let $f : \mathcal{U} \times V \longrightarrow W$ be a bilinear map. Since M is free, we have a map $\tilde{f} : M \longrightarrow W$ s.t. the diag.:

$$\begin{tikzcd} & M \arrow[d, "\tilde{f}"] \\ \mathcal{U} \times V \arrow[ur, "i"] \arrow[r, "f"'] \arrow[r, phantom, bend left=20, "©"] & W \end{tikzcd}$$

commutes.

Since f is bilinear, $\tilde{f}\big|_N \equiv 0$. Let $\bar{f} : M/N \longrightarrow W$ be the map induced by $\tilde{f}$.

So we have:

$$\begin{tikzcd} & M \arrow[r] \arrow[d, "\tilde{f}"] & M/N \arrow[dl, "\bar{f}"] \\ U \times V \arrow[ur, "i"] \arrow[r, "f"'] & W & \end{tikzcd} \qquad \text{i.e.} \qquad \begin{tikzcd} & T \arrow[d, "\bar{f}"] \\ U \times V \arrow[ur, "\tau"] \arrow[r, "f"'] & W \end{tikzcd}.$$

Note that image(τ) generates $T = M/N$ (warning: τ is generally <u>not</u> surjective).

Now $\bar{f}$ is determined by f on every element in image(τ). As image(τ) generates T, we get that $\bar{f}$ is unique. $\square$

<u>Notation.</u> We write $U \otimes_R V$ for T. If R is "clear" from the context we write $U \otimes V$. We write $u \otimes v := \tau(u, v)$, $u \in U$, $v \in V$.

<u>IMPORTANT.</u> $\forall$ $x \in U \otimes V$ can be written as $x = \sum_{i,j} u_i \otimes v_j$ with $u_i \in U$, $v_j \in V$. But, not $\forall$ $x \in U \otimes V$ is of the type $u \otimes v$.

<u>Note:</u> $au \otimes v = u \otimes av$ $\forall$ $u \in U$, $v \in V$, $a \in R$.

$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$ etc.

A few basic properties.

1) $\mathcal{U}\otimes V \cong V\otimes \mathcal{U}$ via a unique iso. σ which satisfies $\sigma(u\otimes v) = v\otimes u$.

2) $(\mathcal{U}\otimes V)\otimes W \cong \mathcal{U}\otimes (V\otimes W)$ via an iso. that satisfies $(u\otimes v)\otimes w \longmapsto u\otimes (v\otimes w)$.

3) $\mathcal{U}\otimes R \cong R\otimes \mathcal{U} \cong \mathcal{U}$.

4) $\left(\bigoplus_{i\in I} \mathcal{U}_i\right)\otimes V \cong \bigoplus_{i\in I} \mathcal{U}_i\otimes V$.

5) If $\mathcal{U}$ & V are free R-mod., then so is $\mathcal{U}\otimes V$ and $\operatorname{rank}(\mathcal{U}\otimes V) = \operatorname{rank}(\mathcal{U})\cdot\operatorname{rank}(V)$.
If $\{u_i\}_{i\in I}$ is a basis of $\mathcal{U}$ & $\{v_j\}_{j\in J}$ is a basis for V, then $\{u_i\otimes v_j\}_{i\in I, j\in J}$ is a basis of $\mathcal{U}\otimes V$.

__Proof.__ 2-3 exc. We'll prove here 1. We <u>cannot</u> just define σ by putting $\sigma(u\otimes v) := v\otimes u$, b.c. it is NOT clear that this def. is good.
Instead, define $\tilde{\sigma}: \mathcal{U}\times V \longrightarrow V\times \mathcal{U}$, $\tilde{\sigma}(u,v) = (v,u)$.

Consider

$$\begin{tikzcd} & & U\otimes V \arrow[d, dashed, "\exists!"', "\bar{\sigma}"] \\ U\times V \arrow[urr, "\tau"] \arrow[r, "\tilde{\sigma}"'] & V\times U \arrow[r, "\tau'"'] & V\otimes U \end{tikzcd}$$

Note that $\tau'\circ\tilde{\sigma}$ is bilinear. $\Rightarrow$ $\exists!$ $\bar{\sigma}: U\otimes V \longrightarrow V\otimes U$ s.t. the diag. above commutes. Clearly $\bar{\sigma}(u\otimes v) = v\otimes u$.

<u>Exc.</u> Show that $\bar{\sigma}$ is an iso. (e.g. by constructing an inverse). $\square$

<u>Induced maps.</u> Let $f: U \longrightarrow U'$, $g: V \longrightarrow V'$ be R-linear maps. Then $\exists!$ a R-linear map, denoted $f\otimes g: U\otimes V \longrightarrow U'\otimes V'$ s.t.

$$\begin{tikzcd} U\times V \arrow[r, "\tau"] \arrow[d, "f\times g"'] & U\otimes V \arrow[d, "f\otimes g"] \\ U'\times V' \arrow[r, "\tau'"'] & U'\otimes V' \end{tikzcd}$$

and $f\otimes g\,(u\otimes v) = f(u)\otimes g(v)$.

Also: $(f_1\circ f_2)\otimes(g_1\circ g_2) = (f_1\otimes g_1)\circ(f_2\otimes g_2)$.

composition of maps.

<u>Exc.</u> Prove this.

Homology with coefficients. Fix an abelian group G.

IMPORTANT: we use additive notation for G.

Let $(C_\cdot, \partial)$ be a chain complex. Define a new ch. complex $(D_\cdot, \tilde{\partial})$, by $D_i := C_i \otimes G$, $\tilde{\partial} := \partial \otimes id$

(All tensor products here are over the ring $\mathbb{Z}$.)

Exc. $(D_\cdot, \tilde{\partial})$ is a ch. complex, i.e. $\tilde{\partial} \circ \tilde{\partial} = 0$.

Notation. $D_\cdot$ is usually denoted by $C_\cdot \otimes G$, and we write $\partial \otimes id$ for $\tilde{\partial}$, but often we just write ∂ again.

We can write elements of $C_k \otimes G$ as $\sum_{i=1}^{l} n_i a_i$ with $l \geq 0$, $n_i \in G$, $a_i \in C_k$ instead of $\sum_{i=1}^{l} n_i \otimes a_i$ or $\sum_{i=1}^{l} a_i \otimes n_i$.

$\partial(\sum n_i a_i) = \sum n_i \partial a_i$.

Remark. C_k is in general not a part of $G \otimes C_k$, so $\nexists$ meaning to take $a \in C_k$ and consider it as an element of $G \otimes C_k$. $\nexists$ to $1 \cdot a$.

We'll apply the above to the ch. complexes of sing. chains $S_\cdot(X)$, $S_\cdot(X,A)$.

And we'll also apply it to the cellular ch. complex $C_\cdot^{cw}(X)$ etc.

$S_\cdot(X;G) := S_\cdot(X)\otimes G$ $\qquad S_n(X;G) = \{ \sum_i n_i \sigma_i : \sigma_i \in \Delta^n \longrightarrow X,\ n_i \in G \}$.

$\partial : S_n(X;G) \longrightarrow S_{n-1}(X;G)$ has the "same" formula

$$\partial(g\sigma) = \sum_{j=0}^{n} (-1)^j \cdot g\, \sigma\Big|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}, \qquad \sigma : \Delta^n \longrightarrow X,\ g\in G.$$

$$\partial\left(\sum n_i\sigma_i\right) := \sum_i n_i\, \partial^{old}(\sigma_i). \qquad \partial^2 = 0. \quad H_n(X;G) := H_n(S_\cdot(X;G)).$$

$A\subset X$ subspace, $\quad S_n(X,A;G) := S_n(X;G)\big/ S_n(A;G) \rightsquigarrow H_n(X,A;G)$.

<u>Reduced homology.</u> $\tilde{H}_n(X;G)$ is the homology of the augmented complex

$$\cdots \xrightarrow{\partial} S_1(X;G) \xrightarrow{\partial} S_0(X;G) \xrightarrow{\varepsilon} G \longrightarrow 0$$

$$\sum n_i x_i \longmapsto \sum n_i \in G$$

Lecture #1B.

Reduced homology: $\tilde{H}_n(X;G)$ is defined as the homology of the augmented complex

$$\cdots \xrightarrow{\partial} S_1(X;G) \xrightarrow{\partial} S_0(X;G) \xrightarrow{\varepsilon} G \longrightarrow 0 \qquad (X \neq \phi)$$

$$\sum n_i x_i \overset{\varepsilon}{\longmapsto} \sum n_i$$

A straightforward argument shows that if X is path-connected $H_0(X;G) \cong G$ via the following iso.: Pick $x_0 \in X$, $G \ni g \longmapsto [g x_0] \in H_0(X;G)$.

This iso is independent of $x_0 \in X$.

<u>Exc.</u> Let $\phi \neq X$ be a space, fix a base pnt $* \in X$. Then $\tilde{H}_0(X;G) \cong H_0(X,*;G)$.

Denote by $\pi_0(X)$ the set of path-connected comp. of X. For $c \in \pi_0(X)$, denote by $X_c \subset X$ the component corresponding to c $\Rightarrow$ $H_0(X;G) \cong \bigoplus_{c \in \pi_0(X)} H_0(X_c;G) \cong \bigoplus_{c \in \pi_0(X)} G$.

Most of the basic homology theory carries over to $H_*(X,A;G)$: LES of a pair, homotopy axiom, Excision, M-V LES.

Calculation.

$$\tilde{H}_k(S^n;G) \cong \begin{cases} G & k=n \\ 0 & k\neq n \end{cases}, \quad \text{i.e. if } n\geq 1 \text{ then } H_k(S^n;G)\cong \begin{cases} 0 & k>n \\ G & k=n \\ 0 & 0<k<n \\ G & k=0 \end{cases}$$

and for $n=0$: $H_k(S^0;G) = \begin{cases} 0 & k\neq 0 \\ G\oplus G & k=0 \end{cases}$.

Proof. For $n=0$, $S^0=\{-1,1\}$ and the result is obvious.

From now on, in the proof we'll omit G from the notation $H_*(-;G)$.

Assume $n\geq 1$. Consider $(B^n, \underbrace{\partial B^n}_{= S^{n-1}})$.

DIGRESSION: Good pairs.

Def. A pair (X,A) is called a good pair if $\phi\neq A\subset X$ is closed & $\exists$ a nbhd. $\mathcal{N}$ of A in X s.t. $A\subset\mathcal{N}$ is a strong defo. retract of $\mathcal{N}$.

Reminder. $A\subset Y$ is a strong defo. retract of Y if $\exists$ a homotopy $F: Y\times I\longrightarrow Y$ s.t. $F(y,0)=y\ \forall y\in Y$, $F(y,1)\in A\ \forall y\in Y$, $F(a,t)=a\ \forall a\in A, t\in I$.

$\Longleftrightarrow$ $\exists$ a retraction $r: Y\to A$ s.t. $(i_A\circ r: Y\to Y) \underset{\text{rel } A}{\simeq} id_Y$.

(inclusion $i_A: A\to Y$)

Important example of a good pair: X = CW complex, $\emptyset \neq A \subset X$ a subcomplex.

__Thm.__ Let (X, A) be a good pair. Consider X/A, and denote by $* \in X/A$ the point corresp. to the point of A. Denote by $q: (X, A) \longrightarrow (X/A, *)$, the quot. map.

Then $q_*: H_k(X, A) \xrightarrow{\cong} H_k(X/A, *) \cong \tilde{H}_k(X/A)$ an iso. $\forall\ k \in \mathbb{Z}$.

Moreover, the statement holds with coeffs.

Assume $n \geq 1$. Consider $(B^n, \underset{= S^{n-1}}{\partial B^n})$. Note that $(B^n, \partial B^n)$ is a good pair.

We'll use now the LES of $(B^n, \partial B^n)$ & the Thm about good pair and get:

$$\cdots \longrightarrow \underset{\overset{\|}{0}}{\tilde{H}_k(B^n)} \longrightarrow \tilde{H}_k(\underbrace{B^n/\partial B^n}_{\approx S^n}) \xrightarrow{\cong} \tilde{H}_{k-1}(\underbrace{\partial B^n}_{= S^{n-1}}) \longrightarrow \underset{\overset{\|}{0}}{\tilde{H}_{k-1}(B^n)}$$

$$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}).$$

$$\Rightarrow \tilde{H}_k(S^n) \cong \cdots \cong \tilde{H}_{k-n}(S^0) = \begin{cases} G & k = n \\ 0 & k \neq n \end{cases}.$$

$\blacksquare$

Q. What are the chain-level representatives of the elements of $\tilde{H}_n(S^n; G)$?

Consider $\sigma_0 : \Delta^n \longrightarrow \Delta^n/\partial\Delta^n$ be the quot. map, viewed here as an n-dim. simplex in the space $\Delta^n/\partial\Delta^n$ $(\approx S^n)$.

Note that σ_0 is an n-cycle in $S_n(\Delta^n/\partial\Delta^n, *)$, where $* \in \Delta^n/\partial\Delta^n$ corresponds to the points of $\partial\Delta^n$.

Exc. 1) Show that $[\sigma_0] \in H_n(\Delta^n/\partial\Delta^n, *) \cong H_n(S^n, *) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$ is a generator.

2) Consider the following map

$$\begin{array}{ccc} G & \longrightarrow & H_n(\Delta^n/\partial\Delta^n, * ; G). \\ \Psi & & \Psi \\ g & \longmapsto & [g\sigma_0] \end{array}$$

show this is an iso.

<u>Change of coeffs.</u> Let $\varphi: G_1 \to G_2$ be a homo. of abelian groups.

$\rightsquigarrow$ a chain map $\varphi^c : S_\cdot(X, A; G_1) \longrightarrow S_\cdot(X, A; G_2)$.

If $f: (X, A) \to (Y, B)$ is a map, then $\exists$ a commut. square:

$$\begin{tikzcd} S_\cdot(X, A; G_1) \arrow[r, "f_c"] \arrow[d, "\varphi^c"'] \arrow[dr, phantom, "\copyright"] & S_\cdot(Y, B; G_1) \arrow[d, "\varphi^c"] \\ S_\cdot(X, A; G_2) \arrow[r, "f_c"] & S_\cdot(Y, B; G_2) \end{tikzcd}$$

exc.

$\Rightarrow$ we get a commut. square in hlgy:

$$\begin{tikzcd} H_*(X, A; G_1) \arrow[r, "f_*"] \arrow[d, "\varphi_*"'] \arrow[dr, phantom, "\copyright"] & H_*(Y, B; G_1) \arrow[d, "\varphi_*"] \\ H_*(X, A; G_2) \arrow[r, "f_*"] & H_*(Y, B; G_2) \end{tikzcd}$$

If $0 \to G' \to G \to G'' \to 0$ is a SES of ab. grps, then we get a SES of ch. complexes

$$0 \to S_\cdot(X, A; G') \longrightarrow S_\cdot(X, A; G) \longrightarrow S_\cdot(X, A; G'') \longrightarrow 0.$$

$\Rightarrow$ We get a LES in hlgy: $\cdots \to H_n(X, A; G') \longrightarrow H_n(X, A; G) \longrightarrow H_n(X, A; G'') \longrightarrow H_{n-1}(X, A; G') \to \cdots$

<u>Two interesting examples:</u>

(*) $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0$

(*) $0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0$

$k \longmapsto k \cdot p$

the connect. homo. here is called the Bockstein homo.

Degree theory with coeffs. in G.

Recall that if $f: S^n \longrightarrow S^n$, then $\deg(f) = d \in \mathbb{Z}$, where $d \in \mathbb{Z}$ is the unique integer s.t.

$$\begin{array}{ccc} \tilde{H}_n(S^n) & \xrightarrow{f_*} & \tilde{H}_n(S^n) \\ \wr\| & & \wr\| \\ \mathbb{Z} & & \mathbb{Z} \end{array} \qquad f_*(a) = d \cdot a.$$

If we take coeffs. in G, we get

$$\begin{array}{ccc} \tilde{H}_n(S^n; G) & \xrightarrow{f_*} & \tilde{H}_n(S^n; G) \\ \wr\| & & \wr\| \\ G & & G \end{array}$$

Q. Can we still say that $f_*(a) = d \cdot a$, with $d \in \mathbb{Z}$, as before?

Yes! If we consider $\sigma_0 : \Delta^n \longrightarrow \Delta^n/\partial\Delta^n \approx S^n$, then we've seen that

$[\sigma_0]$ is a generator of $H_n(S^n, *) \cong \tilde{H}_n(S^n)$. $\Rightarrow$ $f \circ \sigma_0$ is homologous to $d \cdot \sigma_0$ $\Rightarrow$ $f \circ \sigma_0 - d \cdot \sigma_0 = \partial \tau$ for some $\tau : \Delta^{n+1} \longrightarrow S^n$.

$\Rightarrow f_c(g\sigma_0) - d(g\sigma_0) = \partial(g\tau)$ in $S_n(S^n, *; G)$, $\forall g \in G$.

$\Rightarrow f_*[g\sigma_0] = d \cdot [g\sigma_0]$ $\forall g \in G$. $\Rightarrow$ $f_*(a) = d \cdot a$ $\forall a \in \tilde{H}_n(S^n; G)$.

Conclusion: The same recipe for cellular hlgy works also with coeffs. in G.

An interesting example. Consider $\mathbb{R}P^n = S^n / \underset{\forall x \in S^n}{x \sim -x}$.

$\mathbb{R}P^n$ has a CW-complex structure: one cell in each dim. $0 \leq i \leq n$

The $(i-1)$-skeleton of $\mathbb{R}P^n$ is $\mathbb{R}P^{i-1}$. The i'th skeleton $\mathbb{R}P^i$, is obtained

as $\mathbb{R}P^i = \mathbb{R}P^{i-1} \cup_\partial B^i$ with attaching map $f_i : \underset{\underset{S^{i-1}}{\shortparallel}}{\partial B^i} \longrightarrow \mathbb{R}P^{i-1}$, by $f_i(x) := [x]$.

$([x] = [-x])$

The cellular ch. complex $C_\cdot^{cw}(\mathbb{R}P^n)$ has $C_i^{cw} = \mathbb{Z} \cdot e^{(i)}$, $\forall\ 0 \leq i \leq n$.

generator

$\left(\begin{array}{l} \text{and} \\ C_i^{cw} = 0 \\ \forall\ i > n,\ i < 0 \end{array}\right)$

$\partial : C_i^{cw} \longrightarrow C_{i-1}^{cw}$ is $\partial(e^{(i)}) = q_i \cdot e^{(i-1)}$,

where $q_i = \begin{cases} 0 & i > n \\ 0 & i = \text{odd} \\ 2 & i = \text{even} \geq 2 \\ 0 & i \leq 0 \end{cases}$

reminder

$\mathbb{R}P^{i-1} / \mathbb{R}P^{i-2} \approx S^{i-1}$

$f_i \big|_{\partial B^i}$ induces a map $S^{i-1} \longrightarrow S^{i-1}$

q_i is the degree of this map.

deg antipod map $S^{i-1} \to S^{i-1}$ is $(-1)^i$.

$\Rightarrow$ the deg. of the map here is $1 + (-1)^i$.

$\Rightarrow$ For $G = \mathbb{Z}$ we get:

$\circledast$ If n = even:

$H_i(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} 0 & i > n \\ 0 & 2 \leq i = \text{even} \leq n \\ \mathbb{Z}_2 & 1 \leq i = \text{odd} < n \\ \mathbb{Z} & i = 0 \end{cases}$

⊛ If $n = odd$:

$$H_i(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} 0 & i>n \\ \mathbb{Z} & i=n \\ 0 & 2\le i = even \\ \mathbb{Z}_2 & 1\le i = odd < n \\ \mathbb{Z} & i=0 \end{cases}$$

What happens for other G's ? ⟶

$C_i^{cw}(\mathbb{R}P^n; G) = G\cdot e^{(i)}$

$\forall\ 0\le i\le n$, $C_i^{cw}=0$ for $i>n$ & $i<0$,

$\partial: C_i^{cw} \longrightarrow C_{i-1}^{cw}$ is multip. by q as before

1) Assume $\forall\ g\in G$, $\exists!\ h\in G$ s.t. $2h=g$. (e.g. $G=\mathbb{Q}$, $G=\mathbb{R}$, $G=\mathbb{C}$ or any field of char $\neq 2$). $\Rightarrow$ $G \xrightarrow{\times 2} G$ is an iso. $\Rightarrow$ $\partial: C_i^{cw} \xrightarrow{\cong} C_{i-1}^{cw}$ $\forall\ n\ge i = even \ge 2$. $\Rightarrow$ $H_i(\mathbb{R}P^n; G) = 0$ $\forall$ $i = even\ge 2$, $H_0(\mathbb{R}P^n;G)\cong G$, $H_i(\mathbb{R}P^n; G) = 0$ $\forall$ $i = odd < n$. If $n = even$ $H_n(\mathbb{R}P^n;G)=0$ and if $n = odd$, $H_n(\mathbb{R}P^n;G)\cong G$.

Lecture #2A.

cellular homology of $\mathbb{R}P^n$ with coeffs in a group G.

$\mathbb{R}P^n$ has the struct. of a CW complex with one i-cell in each dim. $0 \leq i \leq n$.

Fix an ab. grp. G. $\quad C_i := C_i^{cw}(\mathbb{R}P^n; G) = G \cdot e^{(i)}$ ← symbol denoting the i-cell.

$$d: C_i \longrightarrow C_{i-1}, \quad d(e^{(i)}) = (1+(-1)^i) \cdot e^{(i-1)}.$$

If $\quad 0 < i = \text{even} \leq n \quad \Rightarrow \quad d(e^{(i)}) = 2 \cdot e^{(i-1)} \quad (\Rightarrow d(a \cdot e^{(i)}) = 2a\, e^{(i-1)} \ \forall a \in G).$

If $\quad 0 < i = \text{odd} \leq n \quad \Rightarrow \quad d(e^{(i)}) = 0.$

$$\underset{\substack{\nearrow \\ \text{cycles} \subset C_i}}{Z_i} = \begin{cases} K \cdot e^{(i)} & 0 < i = \text{even} \leq n \\ G \cdot e^{(i)} & 0 < i = \text{odd} \leq n \\ G \cdot e^{(0)} & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}$$

$$K := \ker\left(G \xrightarrow{\times 2} G\right) \subset G$$

$$\underset{\substack{\parallel \\ d(C_{i+1}) \\ \text{boundaries}}}{B_i} = \begin{cases} 0 & i = n \\ 0 & 0 < i = \text{even} < n \\ 2G \cdot e^{(i)} & 0 < i = \text{odd} < n \\ 0 & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}$$

$$2G = \text{image}\left(G \xrightarrow{\times 2} G\right) \subset G.$$

$$H_i^{cw}(\mathbb{R}P^n; G) \cong \begin{cases} K & 0 < i = \text{even} < n \\ G/2G & 0 < i = \text{odd} < n \\ G & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}, \quad H_n^{cw}(\mathbb{R}P^n; G) \cong \begin{cases} K & n = \text{even} \\ G & n = \text{odd}. \end{cases}$$

Several interesting examples of G.

1) $G = \mathbb{Z}$. $\quad 2G = 2\mathbb{Z} \subset \mathbb{Z}$, $K = 0$.

$$H_i(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} 0 & 0 < i = \text{even} < n \\ \mathbb{Z}_2 & 0 < i = \text{odd} < n \\ \mathbb{Z} & i = 0 \\ 0 & i < 0 \text{ or } i > n \end{cases}, \quad H_n(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} 0 & n = \text{even} \\ \mathbb{Z} & n = \text{odd} \end{cases}$$

2) Let G be an ab. grp, s.t. $\forall\, g \in G$, $\exists!\, h \in G$ s.t. $2h = g$ (e.g. $G = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or any field of char $\neq 2$). $\Rightarrow K = 0$, $2G = G$ so $G/2G = 0$.

$$H_0(\mathbb{R}P^n; G) \cong G, \quad H_i(\mathbb{R}P^n; G) = 0 \quad \forall\ 0 < i < n, \quad H_n(\mathbb{R}P^n; G) = \begin{cases} 0 & n = \text{even} \\ G & n = \text{odd}. \end{cases}$$

3) $G = \mathbb{Z}_2$. $2G = 0$, $K = \mathbb{Z}_2$, $G/2G = \mathbb{Z}_2$.

$H_i(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \forall\ 0 \leq i \leq n.$

Application: Borsuk-Ulam Thm.

<u>Thm.</u> Let $f: S^n \to \mathbb{R}^n$ be a contin. map $\Rightarrow$ $\exists\ x \in S^n$ s.t. $f(x) = f(-x)$.

<u>Example.</u> Take $n=2$, S^2 = surface of Earth $f(x) = (\text{temp}_{t_0}(x), \text{press}_{t_0}(x))$.

<u>Preparations for the proof.</u> IMPORTANT to keep in mind: $H_i(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ $\forall\ 0 \leq i \leq n$.

Let $\pi: X \to Y$ be a 2:1 covering.

Let $\Theta: X \to X$ be the unique deck-transf. s.t. $\Theta \neq id$

so $\Theta(x) \neq x$ $\forall x \in X$, $\Theta \circ \Theta = id$.

$$\begin{tikzcd} X \arrow[rr, "\Theta"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

<u>Example.</u> $X = S^n$, $Y = \mathbb{R}P^n = S^n/_{x \sim -x}$. $\Theta(x) = -x$.

We'll work now with $S_\cdot(X; \mathbb{Z}_2)$ and $S_\cdot(Y; \mathbb{Z}_2)$.

Let $\lambda: \Delta^k \longrightarrow X$ be a k-simplex. $\Rightarrow$ $\Theta \circ \lambda$ is a different simplex.

Let $\sigma: \Delta^k \to Y$ —— " —— $\Rightarrow$ σ can be lifted to $\tilde{\sigma}: \Delta^k \longrightarrow X$.

$\exists$ exactly two possible such liftings: $\tilde{\sigma}$ and $\Theta \circ \tilde{\sigma}$.

($\exists$ a lifting b.c. Δ^k is simply connected).

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ \Delta^k \arrow[ur, dashed, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

Define $T: S_{\cdot}(Y; \mathbb{Z}_2) \longrightarrow S_{\cdot}(X; \mathbb{Z}_2)$

$$\sigma \overset{T}{\longmapsto} \tilde{\sigma} + \Theta \circ \tilde{\sigma}$$

(this is independent of the choice of the lift $\tilde{\sigma}$ of σ)

<u>Claim.</u> T is a chain map. <u>Proof.</u> exe.

<u>Claim.</u> T fits into the following SES of ch. complexes: (exe.)

$$0 \longrightarrow S_{\cdot}(Y; \mathbb{Z}_2) \xrightarrow{T} S_{\cdot}(X; \mathbb{Z}_2) \xrightarrow{\pi_c} S_{\cdot}(Y; \mathbb{Z}_2) \longrightarrow 0.$$

For the exactness it is crucial to work with $\mathbb{Z}_2$-coeffs. $\left(\pi_c \circ T(\sigma) = 2\sigma\right)$.

$\Rightarrow$ we get a LES in homology:

$$\cdots \longrightarrow H_k(Y; \mathbb{Z}_2) \xrightarrow{T_*} H_k(X; \mathbb{Z}_2) \xrightarrow{\pi_*} H_k(Y; \mathbb{Z}_2) \xrightarrow{\partial_*} H_{k-1}(Y; \mathbb{Z}_2) \longrightarrow \cdots$$

Suppose we have two coverings each of them 2:1, $X \xrightarrow{\pi} Y$, $X' \xrightarrow{\pi'} Y'$

and we have the deck. transf. $\Theta: X \to X$, $\Theta': X' \to X'$.

Let $f: X \to X'$ be a map s.t. $f \circ \Theta = \Theta' \circ f$. $\Rightarrow$ f descends to $\bar{f}: Y \to Y'$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \pi \downarrow & & \downarrow \pi' \\ Y & \xrightarrow{\bar{f}} & Y' \end{array}$$

We get a map of SES's, induced by f & $\bar{f}$:

$$\begin{array}{ccccccccc} 0 & \to & S_\cdot(Y;\mathbb{Z}_2) & \xrightarrow{T} & S_\cdot(X;\mathbb{Z}_2) & \xrightarrow{\pi_c} & S_\cdot(Y;\mathbb{Z}_2) & \to & 0 \\ & & \bar{f}_c \downarrow & & f_c \downarrow & & \bar{f}_c \downarrow & & \\ 0 & \to & S_\cdot(Y';\mathbb{Z}_2) & \xrightarrow{T'} & S_\cdot(X';\mathbb{Z}_2) & \xrightarrow{\pi'_c} & S_\cdot(Y';\mathbb{Z}_2) & \to & 0 \end{array}$$

exe.: check commut. of the diag.

Take $X = S^n$, $Y = \mathbb{R}P^n$, $X' = S^m$, $Y' = \mathbb{R}P^m$, Θ, Θ' - antipodal maps.

<u>Thm.</u> Let $\phi: S^n \to S^m$ be an odd map (i.e. $\phi(-x) = -\phi(x)$, or equiv. $\phi \circ \Theta = \Theta' \circ \phi$).

Then $n \leq m$.

<u>Proof.</u> Assume by contradiction that $n > m$. Also w.l.o.g. assume $m > 0$, b.c. for $m = 0$ the statement is obvious $\nexists$ odd map $S^n \longrightarrow \underset{\substack{\shortparallel \\ \{-1,1\}}}{S^0}$ if $n > 0$.

Consider
$$\begin{array}{ccc} S^n & \xrightarrow{\ \phi\ } & S^m \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathbb{R}P^n & \xrightarrow{\ \bar{\phi}\ } & \mathbb{R}P^m \end{array}$$
& Consider the LES, discussed before, for $S^m \longrightarrow \mathbb{R}P^m$:

$$\begin{array}{l}
0 \longrightarrow \underset{\cong \mathbb{Z}_2}{H_m(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow[\cong]{T'_*} \underset{\cong \mathbb{Z}_2}{H_m(S^m;\mathbb{Z}_2)} \xrightarrow{\pi'_*} \underset{\cong \mathbb{Z}_2}{H_m(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow[\cong]{\partial_*} \\
\xrightarrow{\cong} \underset{\cong \mathbb{Z}_2}{H_{m-1}(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow{T'_*} \underset{\cong 0}{H_{m-1}(S^m;\mathbb{Z}_2)} \xrightarrow{\pi'_*} \underset{\cong \mathbb{Z}_2}{H_{m-1}(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow[\cong]{\partial_*} \\
\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\longrightarrow \underset{\cong \mathbb{Z}_2}{H_1(\mathbb{R}P^m;\mathbb{Z}_2)} \longrightarrow \underset{\cong 0}{H_1(S^m;\mathbb{Z}_2)} \longrightarrow \underset{\cong \mathbb{Z}_2}{H_1(\mathbb{R}P^m;\mathbb{Z}_2)} \xrightarrow[\cong]{\partial_*} \\
\longrightarrow \underset{\cong \mathbb{Z}_2}{H_0(\mathbb{R}P^m;\mathbb{Z}_2)} \longrightarrow \underset{\cong \mathbb{Z}_2}{H_0(S^m;\mathbb{Z}_2)} \longrightarrow \underset{\cong \mathbb{Z}_2}{H_0(\mathbb{R}P^m;\mathbb{Z}_2)} \longrightarrow 0
\end{array}$$

<u>claim.</u> The upper leftmost map $H_m(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{T'_*} H_m(S^m;\mathbb{Z}_2)$ is an iso.

exc.

<u>claim.</u> $\partial_* : H_k(\mathbb{R}P^m;\mathbb{Z}_2) \longrightarrow H_{k-1}(\mathbb{R}P^m;\mathbb{Z}_2)$ is an iso. $\forall\ 1 \leq k \leq m$.

Of course, the same happens for the seq. associated to $S^n \longrightarrow \mathbb{R}P^n$. (now $1 \leq k \leq n$).

Consider now:

$$\begin{tikzcd}
H_i(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\partial_*"] \arrow[d, "\overline{\phi}_*"'] & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[d, "\overline{\phi}_*"] \\
H_i(\mathbb{R}P^m;\mathbb{Z}_2) \arrow[r, "\partial_*"] & H_{i-1}(\mathbb{R}P^m;\mathbb{Z}_2)
\end{tikzcd}$$

Begin with $i=1$: $\overline{\phi}_*$ on RHS is an iso. $\Rightarrow$ b.c. ∂_*'s are iso.'s we get that $\overline{\phi}_*$ on LHS is also an iso. Applying this argument repeatedly we get that $\overline{\phi}_* : H_i(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_i(\mathbb{R}P^n;\mathbb{Z}_2)$ is an iso. $\forall\ 0 \leq i \leq m$. In partic. $\overline{\phi}_* : H_m(\mathbb{R}P^n;\mathbb{Z}_2) \longrightarrow H_m(\mathbb{R}P^m;\mathbb{Z}_2)$ is an iso.

$$\Rightarrow \quad \begin{tikzcd}
\mathbb{Z}_2 = H_m(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "T_*"] \arrow[d, "\overline{\phi}_*"', "\cong"] \arrow[dr, phantom, "\copyright"] & H_m(S^n;\mathbb{Z}_2) = 0 \arrow[d, "\phi_*"] \\
H_m(\mathbb{R}P^m;\mathbb{Z}_2) \arrow[r, "T'_*", "\cong"'] & H_m(S^m;\mathbb{Z}_2) = \mathbb{Z}_2
\end{tikzcd}$$

(b.c. $0<m<n$)

By the LES from prev. page T'_* is an iso.

Contradiction. $\square$

Proof of the Borsuk-Ulam Thm.

Let $f: S^n \longrightarrow \mathbb{R}^n$. Assume by contradict. that $f(x) \neq f(-x)$ $\forall x \in S^n$.

Define $\phi : S^n \longrightarrow S^{n-1}$, $\quad \phi(x) := \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|} \in S^{n-1}$.

clearly $\phi(-x) = -\phi(x)$. By the prev. Thm. $n \leq n-1$. Contradiction.

$\square$

Lecture #2B.

<u>Thm. (Lusternik-Schnirelmann).</u> Let $A_1, \ldots, A_\ell \subset S^n$ be ℓ closed subsets s.t. $A_1 \cup \cdots \cup A_\ell = S^n$. If $\ell \leq n+1$, then $\exists$ i s.t. A_i contains a pair of antipodal points.

<u>Proof.</u> w.l.o.g assume $\ell = n+1$ (if $\ell < n+1$, put $A_{\ell+1}, \ldots, A_{n+1} = \phi$).

Assume $A_i \cap (-A_i) = \phi$ $\forall$ $1 \leq i \leq n$ and we'll show that $A_{n+1} \cap (-A_{n+1}) \neq \phi$.

<u>Digression.</u> <u>Urysohn Lemma:</u> Let X be a normal space and $C \subset X$ closed, $\mathcal{U}$ an open subset containing C. Then $\exists$ a contin. function $f: X \to [0,1]$ s.t. $f|_C \equiv 0$ and $f|_{X \setminus \mathcal{U}} \equiv 1$.

<u>Reminder.</u>

Hausd. + compact $\Rightarrow$ normal

By the Urysohn lemma $\exists$ a contin. funct. $f_i: X \to [0,1]$ s.t. $f|_{A_i} \equiv 0$ & $f|_{-A_i} \equiv 1$. Take the funct. $f_1, \ldots, f_n$ and define $f: S^n \longrightarrow \mathbb{R}^n$, $f(x) = (f_1(x), \ldots, f_n(x))$.

By Borsuk-Ulam, $\exists$ $x_0 \in S^n$ s.t. $f(x_0) = f(-x_0)$. clearly $x_0 \notin A_i$ $\forall$ $1 \leq i \leq n$ similarly $-x_0 \notin A_i$ $\forall$ $1 \leq i \leq n$. $\Rightarrow$ $x_0, -x_0 \in S^n \setminus (A_1 \cup \ldots \cup A_n) \subset A_{n+1}$. ▨

Proof of the Thm. about good pairs.

To recall:

Def. A pair (X,A) is called a good pair if $\phi \neq A \subset X$ is closed & $\exists$ a nbhd $\mathcal{N}$ of A in X s.t. $A \subset \mathcal{N}$ is a strong def. retract of $\mathcal{N}$.

Recall some examples. 1) $X = B^n$, $A = \partial B^n$.

2) $X =$ CW complex, $\phi \neq A \subset X$ a subcomplex.

Thm. Let (X,A) be a good pair, and let $q : (X,A) \longrightarrow (X/A, *)$ be the quot. map. Let G be an abelian group.

(put corresp. to A under q.)

Then $q_* : H_n(X,A;G) \longrightarrow H_n(X/A, *; G) \cong \tilde{H}_n(X/A; G)$ is an iso. $\forall$ n.

Proof. We'll omit the coeffs. from the notation.

Let $\mathcal{N} \subset X$ be a nbhd. of A s.t. $A \subset \mathcal{N}$ is a strong defo. retract.

Consider $h : (X \setminus A, \mathcal{N} \setminus A) \longrightarrow (X/A \setminus \{*\}, \mathcal{N}/A \setminus \{*\})$ the obvious map.

Exc. h is a homeomorphism.

Consider the following commut. diag.:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "i_*"] \arrow[d, "q_*"'] \arrow[dr, phantom, "\copyright"] & H_n(X,\mathcal{N}) \arrow[d, "q_*"'] \arrow[dr, phantom, "\copyright"] & H_n(X\setminus A, \mathcal{N}\setminus A) \arrow[l, "\cong"', "\text{exc.}"] \arrow[d, "h_*", "\cong"'] \\
H_n(X/A, *) \arrow[r, "i'_*"] & H_n(X/A, \mathcal{N}/A) & H_n(X/A\setminus\{*\}, \mathcal{N}/A\setminus\{*\}) \arrow[l, "\cong"', "\text{exc.}"]
\end{tikzcd}$$

$\Rightarrow$ q_* in the middle is an iso.

<u>claim.</u> i_* & i'_* are iso.'s.

<u>Proof.</u> Consider the LES of (X,A) and $(X,\mathcal{N})$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "i_{A*}"'] & H_n(X) \arrow[r] \arrow[d, "id"', "\cong"] & H_n(X,A) \arrow[r] \arrow[d, "i_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "i_{A*}"] & H_{n-1}(X) \arrow[r] \arrow[d, "id"', "\cong"] & \cdots \\
\cdots \arrow[r] & H_n(\mathcal{N}) \arrow[r] & H_n(X) \arrow[r] & H_n(X,\mathcal{N}) \arrow[r] & H_{n-1}(\mathcal{N}) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

Note that i_{A*} = iso. b.c. $i_A : A \to \mathcal{N}$ is a homotopy equiv.

By the 5-lemma i_* is also an iso. Similarly, i'_* is an iso. because $\{*\} \longrightarrow \mathcal{N}/A$ is a homotopy equiv. (this follows from $A \subset \mathcal{N}$ being a strong defo. retract). $\blacksquare$

Cohomology.

Algebra. cochain complex.

$$\cdots \to C^{i-1} \xrightarrow{\delta} C^{i} \xrightarrow{\delta} C^{i+1} \to \cdots \qquad \delta \circ \delta = 0$$

cohomology. $H^i(C^\cdot) := \ker\left(C^i \xrightarrow{\delta} C^{i+1}\right) \Big/ \operatorname{image}\left(C^{i-1} \xrightarrow{\delta} C^i\right)$.

cocycles

coboundaries.

Cochain maps. $f : C^\cdot \to \mathcal{D}^\cdot$ s.t. $f \circ \delta_C = \delta_{\mathcal{D}} \circ f$.

$\rightsquigarrow$ f induces a map in cohomology: $f^* : H^*(C^\cdot) \longrightarrow H^*(\mathcal{D}^\cdot)$.

⊛ SES of cochain complexes $\Longrightarrow$ LES in cohomology

$0 \to A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot \to 0$ SES of cochain complexes

$\rightsquigarrow$ a LES in cohlgy: $\cdots \to H^i(A^\cdot) \xrightarrow{f^*} H^i(B^\cdot) \xrightarrow{g^*} H^i(C^\cdot) \xrightarrow{\delta^*} H^{i+1}(A^\cdot) \to \cdots$

⊛ cochain homotopy:

$$\begin{tikzcd} \cdots \arrow[r] & C^{i-1} \arrow[r] & C^{i} \arrow[r] \arrow[dl, "h"'] & C^{i+1} \arrow[r] \arrow[dl, "h"'] & \cdots \\ \cdots \arrow[r] & D^{i-1} \arrow[r] & D^{i} \arrow[r] & D^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

We say that coch. maps $f, g : C^\cdot \to D^\cdot$ are homotopic if $\exists\, h : C^\cdot \to D^{\cdot - 1}$ s.t. $h \circ \delta_C + \delta_D \circ h = f - g$.

If f & g are coch. homotopic $\Rightarrow$ $f^* = g^* : H^*(C^\cdot) \longrightarrow H^*(D^\cdot)$.

<u>Remarks.</u> If $(C_\cdot, d)$ is a chain complex, then $(D^i := C_{-i}, \delta = d)$ is a cochain complex.

<u>Another method.</u> Let $(C_\cdot, \partial)$ be a ch. complex, and let's fix an abelian group G. $D^k := \hom(C_k, G)$. Define $\delta : D^k \longrightarrow D^{k+1}$, $\delta := \partial^*$,

$\delta(f) := f \circ \partial$, $\forall f \in \hom(C_k, G)$.

$$\begin{tikzcd} C_{k+1} \arrow[r, "\partial"] \arrow[dr, "\delta(f)"'] & C_k \arrow[d, "f"] \\ & G \end{tikzcd}$$

We have $\delta^2(f) = \delta(f \circ \partial) = f \circ \partial \circ \partial = 0$.

We get a cochain complex $(D^\cdot, \delta) \rightsquigarrow H^*(D^\cdot, \delta)$.

<u>Rem.</u> 1) $f : C_k \longrightarrow G$ is a cocycle $\Leftrightarrow$ $f\big|_{\partial(C_{k+1})} \equiv 0$, i.e. $f\big|_{B_k} \equiv 0$.

2) If $f =$ coboundary, i.e. $f = \delta(g)$ (i.e. $f = g \circ \partial$ for some g)

$\Rightarrow f\big|_{Z_k} \equiv 0$.

A bit about cohomology of top. spaces.

X space, $A \subset X$ subspace. G = ab. group.

$S^k(X;G) := \hom(S_k(X), G)$, $S^k(X,A;G) := \hom(S_k(X,A), G)$

take $\delta = \partial^*$, as before. $\rightsquigarrow H^*(X;G)$ and $H^*(X,A;G)$. ⟵ singular cohomology.

If $G = \mathbb{Z}$ we write $H^*(X)$ etc.

A cochain $\varphi \in S^k(X;G)$ assigns an element in G to every simplex $\sigma : \Delta^k \to X$: $\varphi(\sigma) \in G$, or $\langle \varphi, \sigma \rangle \in G$.

$$\langle \delta\varphi, \tau \rangle := \langle \varphi, \partial\tau \rangle = \sum_{i=0}^{k+1} (-1)^i \langle \varphi, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+1}]} \rangle . \quad \left(\tau : \Delta^{k+1} \longrightarrow X \right)$$

Examples. $H^0(X;G)$. $\varphi \in S^0(X;G)$, i.e. $\varphi : X \longrightarrow G$.

$$S^0(X;G) \xrightarrow{\delta} S^1(X;G). \quad \langle \delta\varphi, \sigma \rangle = \langle \varphi, \partial\sigma \rangle = \langle \varphi, \sigma(v_1) - \sigma(v_0) \rangle =$$

(a sing. simplex $\Delta^1 \to X$) $\sigma : [v_0, v_1] \to X$

$$= \varphi(\sigma(v_1)) - \varphi(\sigma(v_0)).$$

$\delta\varphi = 0 \iff \varphi$ is const. on each path-connect. component of X.

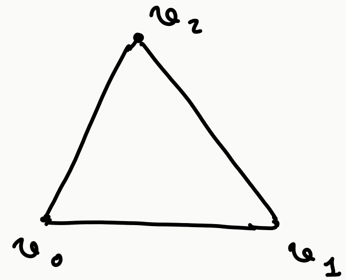
$$\Rightarrow \quad H^0(X;G) \cong \prod_{c \in \pi_0(X)} G$$

$\left(\text{Recall} : H_0(X;G) = \bigoplus_{c \in \pi_0(X)} G \right)$

<u>cocycles in degree 1.</u> $\varphi \in S^1(X;G)$, $\varphi : S_1(X) \longrightarrow G$, so φ assigns an element in G to every path τ in X.

Let $\sigma : \Delta^2 \longrightarrow X$, $\langle \delta\varphi, \sigma \rangle = \langle \varphi, \partial\sigma \rangle = \langle \varphi, \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} \rangle$

So $\delta\varphi = 0 \iff \varphi(\sigma|_{[v_0,v_1]}) + \varphi(\sigma|_{[v_1,v_2]}) = \varphi(\sigma|_{[v_0,v_2]})$.



Lecture #3A

Let $C_\bullet$ & $D_\bullet$ be chain complexes, $\varphi : C_\bullet \to D_\bullet$ a chain map. G = abelian group.

$\rightsquigarrow$ $\varphi^* : \hom(D_\bullet, G) \longrightarrow \hom(C_\bullet, G)$ is a cochain map. $\rightsquigarrow$ it induces a map in cohomology. $\left(\varphi^*(\alpha) = \alpha \circ \varphi \quad \forall \quad \alpha : D_k \longrightarrow G \right)$

Important example from topology: let $f : X \to Y$ be a map between spaces.

$\rightsquigarrow$ $f_c : S_\bullet(X) \longrightarrow S_\bullet(Y)$ chain map. $\rightsquigarrow$ $f_c^* : S^\bullet(Y; G) \longrightarrow S^\bullet(X; G)$ coch. map.

$\rightsquigarrow$ $f^* : H^*(Y; G) \longrightarrow H^*(X; G)$.

<u>Special case:</u> $A \subset X$ subspace, $i : A \to X$ the inclusion. What is i_c^*?

If $\alpha : S_k(X) \longrightarrow G$, then $i_c^*(\alpha)$ is just $\alpha\big|_{S_k(A)} : S_k(A) \longrightarrow G$.

The Universal coeff. Thm.

Split exact sequences. R = comm. ring (with a unit).

Def. Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a SES of R-modules.
We say that the seq. splits if $\exists$ iso. $k: V \xrightarrow{\cong} U \oplus W$ s.t. the diag. commutes:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, equal] & V \arrow[r, "g"] \arrow[d, "k", "\cong"'] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "i"] & U \oplus W \arrow[r, "p"] & W \arrow[r] & 0
\end{tikzcd}$$

$i(u) := (u, 0)$, $p(u, w) := w$

Proposition. A SES $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ splits iff one of the following holds:

1) $\exists$ a homo. $V \xleftarrow{s} W$ s.t. $g \circ s = id_W$.

2) $\exists$ a homo. $U \xleftarrow{\pi} V$ s.t. $\pi \circ f = id_U$.

Proof. Exc.

Example. $R = \mathbb{Z}$. The seq. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ does NOT split.

Prop. Let W be a free R-module. Then $\forall$ SES of R-modules
$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ splits. (exc.)

Exactness & hom. Let M be an R-module. Consider $\hom_R(-, M)$.

Q. Does $\hom_R(-, M)$ preserve exactness of SES's?

> Notation: we'll abbreviate here $\hom := \hom_R$

Prop. If $\quad U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0 \quad$ is an ex.-seq. then

$$\hom(U, M) \xleftarrow{f^*} \hom(V, M) \xleftarrow{g^*} \hom(W, M) \longleftarrow 0$$

is also exact.

But. If $\quad 0 \to U \xrightarrow{f} V \quad$ is exact (i.e. $f : U \longrightarrow V$ is injective)

then $\quad 0 \longleftarrow \hom(U, M) \xleftarrow{f^*} \hom(V, M) \quad$ might NOT be exact (i.e. f^* might NOT be surjective).

For example. $R = \mathbb{Z}$. $\quad 0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \quad (m \neq 1, -1, 0)$. $\Longrightarrow$ $\hom(\mathbb{Z}, M) \longleftarrow \hom(\mathbb{Z}, M)$

$$\begin{array}{ccc} & & \rotatebox{90}{$\in$} \\ m \cdot \varphi & \longleftarrow\!\shortmid & \varphi \end{array}$$

Conclusion. $\hom(-, M)$ does NOT preserve SES.

Prop. If $0 \to U \to V \to W \to 0$ is a split SES, then $\forall$ R-mod. M

$$0 \leftarrow \hom(U,M) \leftarrow \hom(V,M) \leftarrow \hom(W,M) \leftarrow 0$$ is exact and moreover this seq. is also split.

Outline of proof. w.l.o.g. we may assume that our original seq. is

$$0 \to U \xrightarrow{i} U \oplus W \xrightarrow{p} W \to 0 \quad \text{with} \quad i(u) = (u,0), \quad p(u,w) = w.$$

Exc. Justify this step.

$$\begin{tikzcd}
0 & \hom(U,M) \arrow[l] & \hom(U\oplus W, M) \arrow[l] \arrow[d, equal] & \hom(W,M) \arrow[l] & 0 \arrow[l] \\
& & \hom(U,M)\oplus\hom(W,M) \arrow[ul, "q"] & &
\end{tikzcd}$$

$$q(\sigma, \tau) = \sigma$$

we have exactness here b.c. q is surjective.

∎

From now on, $R = \mathbb{Z}$, so we work with abelian groups.

Let $(C_\bullet, \partial)$ be a chain complex of free abelian groups. Fix an ab. grp. G.

Consider $(C^*, \delta) = \left(\hom(C_\bullet, G), \partial^*\right)$ as a cochain complex. Denote the cohomology of the latter by $H^*(C; G)$. What is the relation between $H^*(C;G)$ & $H_*(C_\bullet)$?

Claim. $\exists$ an obvious map $h : H^n(C;G) \longrightarrow \hom(H_n(C), G)$ $\forall n$ which is surjective.

Proof. Put $Z_n := \ker \partial \subset C_n$, $B_n := \partial(C_{n+1})$. A class $\alpha \in H^n(C;G)$ is represented by $\varphi : C_n \longrightarrow G$ s.t. $\varphi \circ \partial = 0$, i.e. $\varphi|_{B_n} \equiv 0$. $\Rightarrow \varphi$ descends to $\bar\varphi : \underbrace{Z_n/B_n}_{\boxed{= H_n(C)}} \longrightarrow G$. Define $h(\alpha) := \bar\varphi$.

Note that the def. of h is good, since if $[\varphi'] = \alpha$, then $\varphi - \varphi' = \psi \circ \partial$ for some $\psi : C_{n-1} \longrightarrow G$.

$\Rightarrow \varphi - \varphi' \equiv 0$ on Z_n $\Rightarrow$ $\overline{\varphi'} = \bar\varphi$.

Exc. h is linear.

Claim. h is surjective.

<u>Proof.</u> We'll construct a right-inverse to h,

$$s : \hom(H_n(C), G) \longrightarrow H^n(C; G) \qquad (\& \ h \circ s = id).$$

Consider the SES $\quad 0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0.$

B_{n-1} is a subgroup of C_{n-1} and C_{n-1} is free (by assumption).

$\Rightarrow B_{n-1}$ is also free. $\Rightarrow$ The SES splits. $\Rightarrow \exists \ Z_n \xleftarrow{p} C_n$

a left-inverse of i, i.e. $p \circ i = id_{Z_n}$. $\Rightarrow \forall$ homo. $\varphi_0 : Z_n \longrightarrow G$

we can define an extension $\varphi := \varphi_0 \circ p : C_n \longrightarrow G$ s.t. $\varphi\big|_{Z_n} = \varphi_0$.

The resulting map $p^* : \hom(Z_n, G) \longrightarrow \hom(C_n, G)$ is a homo.

Now let $\vartheta \in \hom(H_n(C), G)$. $\Rightarrow \ \vartheta : Z_n/B_n \longrightarrow G$.

Put $\vartheta' := \left(Z_n \longrightarrow Z_n/B_n \xrightarrow{\vartheta} G \right)$. Define $\hat{\vartheta} := p^*(\vartheta') = \vartheta' \circ p : C_n \longrightarrow G$.

We have $\hat{\vartheta} \circ \partial = \vartheta' \circ p \circ \partial = \vartheta' \circ \partial = 0. \ \Rightarrow \ \delta(\hat{\vartheta}) = 0.$ Define $s(\vartheta) := [\hat{\vartheta}]$.

$p\big|_{Z_n} = id$

Easy to check that s is linear (exc.)

Also $h \circ s(\vartheta) = h([\hat{\vartheta}]) = \vartheta$. (exc.) ∎

Conclusion. h fits into a SES

$$0 \longrightarrow \ker(h) \longrightarrow H^n(C;G) \xrightarrow{h} \hom(H_n(C),G) \longrightarrow 0$$

which is split, (b.c. we've seen $\exists$ a right-inverse s to h).

Example.
$$C_\bullet = \Big(0 \to \underset{C_3}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{0} \underset{C_2}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{\times 2} \underset{C_1}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{0} \underset{C_0}{\overset{\shortparallel}{\mathbb{Z}}} \to 0 \Big)$$
(This is the cellular ch. complex of $\mathbb{R}P^3$)

$H_0(C_\bullet) = \mathbb{Z}$, $H_1(C_\bullet) = \mathbb{Z}_2$, $H_2(C_\bullet) = 0$, $H_3(C_\bullet) = \mathbb{Z}$.

Take $G = \mathbb{Z}$.
$$C^* = \Big(0 \leftarrow \underset{C_3^*}{\overset{\shortparallel}{\mathbb{Z}}} \xleftarrow{0} \underset{C_2^*}{\overset{\shortparallel}{\mathbb{Z}}} \xleftarrow{\times 2} \underset{C_1^*}{\overset{\shortparallel}{\mathbb{Z}}} \xleftarrow{0} \underset{C_0^*}{\overset{\shortparallel}{\mathbb{Z}}} \leftarrow 0 \Big)$$

$H^0(C^*) = \mathbb{Z}$, $H^1(C^*) = 0$, $H^2(C^*) = \mathbb{Z}_2$, $H^3(C^*) = \mathbb{Z}$. So $h: H^2(C^*) \longrightarrow \hom(H_2(C),\mathbb{Z})$ has a kernel

We'll use the notation: for an abelian grp E, write $E^* := \hom(E, G)$.

Goal. Understand better $\ker h$.

Consider the commut. diag.

$$\begin{tikzcd} 0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

The rows are SES.

Since C_n & C_{n-1} are free, so are B_n & B_{n-1}. $\Rightarrow$ The two rows are split $\Rightarrow$ after applying $\hom(-, G)$ we get

$$\begin{tikzcd} 0 & Z_{n+1}^* \arrow[l] & C_{n+1}^* \arrow[l, "j^*"'] & B_n^* \arrow[l, "\delta"'] & 0 \arrow[l] \\ 0 & Z_n^* \arrow[l] \arrow[u, "0"] & C_n^* \arrow[l, "j^*"'] \arrow[u, "\delta"] & B_{n-1}^* \arrow[l, "\delta"] \arrow[u, "0"'] & 0 \arrow[l] \end{tikzcd}$$

with exact rows.

View $Z_\cdot^*$ & $B_\cdot^*$ as cochain complexes with 0-differentials. $\Rightarrow$ the last diag. is actually a SES of cochain complexes. $\Rightarrow$ we get a LES in cohomology:

$$\cdots \leftarrow B_n^* \xleftarrow{\tau} Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow{\tau} Z_{n-1}^* \leftarrow \cdots$$

connect. homo.   connect. homo.

$$\begin{array}{ccccccccc}
0 & \leftarrow & Z_{n+1}^* & \xleftarrow{j^*} & C_{n+1}^* & \xleftarrow{\delta} & B_n^* & \leftarrow & 0 \\
 & & 0\uparrow & & \delta\uparrow & & \uparrow 0 & & \\
0 & \leftarrow & Z_n^* & \xleftarrow{j^*} & C_n^* & \xleftarrow[\delta]{} & B_{n-1}^* & \leftarrow & 0
\end{array}$$

<u>claim.</u> τ (the connect. homo) is just i^*, where $i: B_\cdot \longrightarrow Z_\cdot$ is the inclusion.

In other words τ is just the restriction map. (exe.)

Lecture #3B.

h fits into a SES $0 \to \ker(h) \to H^n(C;G) \xrightarrow{h} \hom(H_n(C),G) \to 0$ which is split, (b.c. we've seen $\exists$ a right-inverse s to h).

Goal. Understand better $\ker h$.

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

after applying $\hom(-,G)$ we get

$$\begin{tikzcd}
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l, "j^*"'] & B^*_n \arrow[l, "\delta"'] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l, "j^*"'] \arrow[u, "\delta"] & B^*_{n-1} \arrow[l, "\delta"] \arrow[u, "0"'] & 0 \arrow[l]
\end{tikzcd}$$

In cohomology we get the LES: $\cdots \leftarrow B^*_n \xleftarrow{\tau} Z^*_n \xleftarrow{j^*} H^n(C;G) \leftarrow B^*_{n-1} \xleftarrow{\tau} Z^*_{n-1} \leftarrow \cdots$

connect. homo.

connect. homo.

claim. τ (the connect. homo) is just i^*, where $i: B_\bullet \to Z_\bullet$ is the inclusion.

In other words τ is just the restriction map. (exe.)

Denote by $i_n : B_n \longrightarrow Z_n$ the inclusion. Take now the previous LES & split into many SES's:

$$\cdots \leftarrow B_n^* \xleftarrow[i_n^*]{\tau} Z_n^* \xleftarrow{j^*} H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow[i_{n-1}^*]{\tau} Z_{n-1}^* \leftarrow \cdots$$

We get: $\quad 0 \leftarrow \ker(i_n^*) \xleftarrow{\tilde{j}} H^n(C;G) \longleftarrow \operatorname{coker}(i_{n-1}^*) \longleftarrow 0$

Now $\ker(i_n^*) \cong \hom(H_n(C);G)$. Indeed, if $\varphi : Z_n \longrightarrow G$ s.t. $\varphi|_{B_n} \equiv 0$

$\Rightarrow \bar{\varphi} : H_n(C) = Z_n/B_n \longrightarrow G$. And vice-versa, if $\bar{\varphi} : H_n(C) \longrightarrow G$ then we can define $\varphi = \left(Z_n \longrightarrow Z_n/B_n \xrightarrow{\bar{\varphi}} G\right)$ and clearly $i_n^*(\varphi) = 0$.

Denote this iso. by $\Theta : \ker(i_n^*) \longrightarrow \hom(H_n(C), G)$.

<u>Claim.</u> The following diag. comm.

$$\begin{tikzcd}
\ker(i_n^*) \arrow[d, "\Theta"', "\cong"] & H^n(C;G) \arrow[l, "\tilde{j}"'] \arrow[dl, "h"] \\
\hom(H_n(C), G) &
\end{tikzcd}$$

(exc.)

we deduce that we have a split SES:

$$0 \to \mathrm{coker}(i_{n-1}^*) \longrightarrow H^n(C;G) \xrightarrow{h} \hom(H_n(C),G) \longrightarrow 0.$$

Consider the SES: $0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0$

<u>Dualize it:</u> $B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \longleftarrow \hom(H_{n-1}(C),G) \longleftarrow 0$

complete it to an ex. seq.

$$0 \leftarrow \mathrm{coker}(i_{n-1}^*) \longleftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \longleftarrow \hom(H_{n-1}(C),G) \longleftarrow 0 \qquad (*)$$

We'll see soon that $\mathrm{coker}(i_{n-1}^*)$ depends only on $H_{n-1}(C)$ & G.

Resolutions. Fix an ab. group H. Sometimes we'll view H as a chain complex concentrated at deg. 0 : $\cdots \to 0 \to 0 \to H \to 0 \to 0 \to \cdots$
we'll denote this ch complex also by H.

Def. A free resolution of H is a chain complex $F_\bullet$ with degrees ≥ 0 $\left(\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0 \right)$ together with a map $F_0 \xrightarrow{\varepsilon} H$ s.t.

1) F_i is free ab. grp $\forall i$.
2) The seq. $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{\varepsilon} H \to 0$ is exact.
(i.e. the ch. complex $\nearrow$ has 0 homology).

We denote it by $F_\bullet \xrightarrow{\varepsilon} H$.

Exc. $F_\bullet \xrightarrow{\varepsilon} H$ is a free resolution iff: $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0$ is a ch. complex & the vert. map is a chain map that induces an iso. in homology. (i.e. the vertic. map is a quasi-iso.)

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d] & F_1 \arrow[r, "f_1"] \arrow[d] & F_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[d] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

Given a free resolution of H, apply to it $\hom(-,G)$:

we obtain $\quad \cdots \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \leftarrow 0$

Note that $\quad \cdots \leftarrow F_2^* \leftarrow \underbrace{F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0}_{\text{exact.}}$

but the entire coch. complex might not be everywhere acyclic (i.e. exact).

Consider the cohomology of the 1'st seq. $F_\cdot^*$

Denote it by $H^n(F;G)$.

$\quad \parallel \quad \ker f_{n+1}^* / \operatorname{image} f_n^*.$

__Exc.__ $\quad H^0(F;G) \cong H^* = \hom(H,G)$.

__Remark.__ Recall the seq. from the previous discussion

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0.$$

This is a free resolution of the group $H := H_{n-1}(C)$.

$\underbrace{\cdots \to 0 \to B_{n-1} \longrightarrow Z_{n-1}}$ $\quad (F_0 = Z_{n-1},\ F_1 = B_{n-1},\ F_i = 0\ \forall\ i \geq 2).$

We get after dualizing $\quad \cdots 0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \hom(H_{n-1}(C), G) \leftarrow 0$

Note that $\quad \operatorname{coker}(i_{n-1}^*) = H^1(F;G)$.

Main Lemma. 1) Let $F_\cdot \xrightarrow{\varepsilon} H$ be a free resol. of H and $F'_\cdot \xrightarrow{\varepsilon'} H'$ a resolut. (not necess. free) of H'. Then every homo. $\alpha : H \to H'$ can be extended to a chain map $F_\cdot \to F'_\cdot$, i.e.

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "\varepsilon"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\ \cdots \arrow[r] & F'_2 \arrow[r, "f'_2"'] & F'_1 \arrow[r, "f'_1"'] & F'_0 \arrow[r, "\varepsilon'"] & H' \arrow[r] & 0 \end{tikzcd}$$

Moreover, every two such extensions are ch. homotopic.

2) For every two free resolut. $F_\cdot$ & $F'_\cdot$ of H, $\exists$ canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$ $\forall\, n \geq 0$. In other words $H^n(F;G)$, $n = 0, 1, 2, \dots$ depend only on H & G (and NOT on the choice of $F_\cdot$).

Main point. Every ab. group H has a free resolut. of the type

$\cdots \to 0 \to F_1 \to F_0 \xrightarrow{\varepsilon} H \to 0$, i.e. with $F_i = 0$ $\forall\, i \geq 2$. Indeed, choose a set of generators S for H. Let $F_0 := \bigoplus_{s\in S} \mathbb{Z}\cdot x_s$, where x_s is a symbol corresp. to $s \in S$. (i.e. F_0 is the free ab. group on the set S).

We have a surj. homo $F_0 \xrightarrow{\varepsilon} H$, $\varepsilon(x_s) := s$. Take $F_1 := \ker(\varepsilon)$.

$F_1 \subset F_0$ is a subgroup $\Rightarrow$ F_1 is also free. We get a SES

$$0 \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\varepsilon} H \to 0.$$

Conclusion. For every free resolut. $F_\bullet \to H$ of H we have $H^i(F;G) = 0$ $\forall\, i \geq 2$. (This follows from the prev. lemma + the prev. short resolut.).

The only two interesting groups are $H^0(F;G)$ & $H^1(F;G)$.

We've seen $H^0(F;G) = \hom(H,G)$.

Notation: $\operatorname{Ext}(H,G) := H^1(F;G)$.

Thm. (The universal coeffs. Thm.) Let $C_\bullet$ be a chain complex of free ab. grps. Let G be an ab. group. Then $\exists$ a split SES:

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \hom(H_n(C), G) \longrightarrow 0.$$

Rem. In general $\nexists$ canonical splitting.

How to calculate $\mathrm{Ext}(H, G)$?

Prop. 1) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.

2) If H is a free ab. group then $\mathrm{Ext}(H, G) = 0$ $\forall$ groups G.

3) $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$ $\quad (nG = \{ng : g \in G\} \subset G)$.

Rem. The above 3 statements are enough in order to calculate $\mathrm{Ext}(H, G)$ for all finitely generated ab. groups H. This is because we have a SES

$$0 \to \underbrace{H_{\text{torsion}}}_{\{h \in H : \exists k \in \mathbb{Z} \text{ s.t. } k \cdot h = 0\}} \xrightarrow{\text{inc.}} H \longrightarrow \underbrace{H/H_{\text{torsion}}}_{H_{\text{free}} \text{ - free ab. group.}} \longrightarrow 0.$$

Since H is f. gener., H_{free} is a free ab. group, so the seq. splits and we have

$$H \cong H_{free} \oplus H_{torsion} \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\times r} \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_{k_j}$$

$r = \operatorname{rank}(H_{free})$

$\ell \geq 0$

$2 \leq k_j \in \mathbb{Z}$.

$$\operatorname{Ext}(H, G) \cong \operatorname{Ext}(H_{tors.}, G) \cong \bigoplus_{j=1}^{\ell} \underbrace{\operatorname{Ext}(\mathbb{Z}_{k_j}, G)}_{G/k_j G}.$$

Last lecture: We defined $\mathrm{Ext}(H,G)$ using free resolution of H.

$F_\bullet \to H$ free resolution of H. Write $F_i^* := \hom(F_i, G)$. $\mathrm{Ext}(H,G) := H^1(F^*)$.

(independent upto canonical iso. of the free resolut. $F_\bullet$ of H)

Universal coeff. Thm. Let $C_\bullet$ be a chain complex of free ab. groups. Let G be an ab. grp. Then $\exists$ a split SES

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C;G) \xrightarrow{h} \hom(H_n(C), G) \longrightarrow 0.$$

However, in general $\nexists$ canonical splitting.

Calculation of Ext.

Prop. 1) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$.

2) If H is a free ab. grp then $\mathrm{Ext}(H,G) = 0$.

3) $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$. (In partic. $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$.)

Proof. 1) Let $F_\bullet \to H$ be a free resolut. of H & $F'_\bullet \to H'$ a free resolut. of H'.

$\Rightarrow$ $F_\bullet \oplus F'_\bullet$ is a free resolut. of $H \oplus H'$.

$$\begin{array}{ccccccccc}
\cdots \to & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{\varepsilon} & H \\
& \oplus & & \oplus & & \oplus & & \oplus \\
\cdots \to & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{\varepsilon'} & H'
\end{array}$$

$(F_i \oplus F'_i)^* \cong F_i^* \oplus F_i'^*$

$g_i := f_i \oplus f'_i \Rightarrow g_i^* = f_i^* \oplus f_i'^*$.

$$\operatorname{Ext}(H \oplus H', G) = H^1(F_\cdot \oplus F'_\cdot ; G) \cong H^1(F_\cdot ; G) \oplus H^1(F'_\cdot ; G) = \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G).$$

2) If H is free, then
$$\begin{array}{ccccccccc} \cdots \to & 0 & \to & 0 & \to & H & \xrightarrow{id} & H \\ & \nearrow & & \nearrow & & \nearrow & & \\ F_2 & & F_1 & & F_0 & & & \end{array}$$
is a <u>free</u> resolut. of H.

$$\operatorname{Ext}(H, G) = H^1(F; G) = 0.$$

3) Consider the following resolution of $\mathbb{Z}_n$:

$$\cdots 0 \to 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{q} \mathbb{Z}_n \to 0 \,, \qquad i(a) := na, \quad q := \text{quot. map.}$$

Apply $\hom(-, G)$:
$$\begin{array}{ccccc} & & \text{"}F_1^*\text{"} & & \text{"}F_0^*\text{"} \\ 0 & \leftarrow & \hom(\mathbb{Z}, G) & \xleftarrow{i^*} & \hom(\mathbb{Z}, G) \\ & & \cong & & \cong \\ 0 & \longleftarrow & G & \xleftarrow{\times n} & G \end{array}$$

$$\Rightarrow \operatorname{Ext}(\mathbb{Z}_n, G) = H^1(F^*) = G/nG.$$

$\blacksquare$

<u>Cor.</u> Let $C_\cdot$ be a ch. complex of free ab. groups. If $H_n(C_\cdot)$ & $H_{n-1}(C_\cdot)$ are finitely generated. Denote by $T_n \subset H_n(C_\cdot)$, $T_{n-1} \subset H_{n-1}(C_\cdot)$ the torsion subgroups. Then $H^n(C; \mathbb{Z}) \cong H_n(C_\cdot)_{\text{free}} \oplus T_{n-1}$. ~~However~~
But the splitting is not canonical.

<u>Conclusion.</u> Assume that $H_0(C.)$ is free. Then $H^1(C;\mathbb{Z})$ is free.

The seq. $0 \to \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C;G) \longrightarrow \hom(H_n(C), G) \longrightarrow 0$
is natural w.r.t. chain maps (& homo's of G) in the following sense:
if $\alpha: C. \to C'.$ is a ch. map. Then we have a commut. diag.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C), G) \arrow[r] & H^n(C;G) \arrow[r] & \hom(H_n(C), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C'), G) \arrow[r] \arrow[u, "\alpha_h^{ext}"'] & H^n(C';G) \arrow[r] \arrow[u, "\alpha^*"'] & \hom(H_n(C'), G) \arrow[r] \arrow[u, "\alpha_h^*"'] & 0
\end{tikzcd}$$

exc.: prove this

$\alpha_h: H_n(C) \to H_n(C')$, α_h^* is the dual of α_h.

map induced by α

α^* = the map induced in cohomology from α.

α_h^{ext} = map on Ext induced by α_h.

<u>Example from topology.</u> $X = \mathbb{R}P^n$, $G = \mathbb{Z}_2$.

$$0 \longrightarrow \mathrm{Ext}(H_{i-1}(\mathbb{R}P^n), \mathbb{Z}_2) \longrightarrow H^i(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow \hom(H_i(\mathbb{R}P^n), \mathbb{Z}_2) \longrightarrow 0$$

Assume $n = \text{even} > 0$. $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2$, $H_2(X) = 0, \ldots, H_{2k-1}(X) = \mathbb{Z}_2$, $H_{2k}(X) = 0$, $1 \leq 2k-1 \leq n-1$

$\ldots, H_n(X) = 0$.

$$\mathrm{Ext}(H_j(X), \mathbb{Z}_2) = \begin{cases} 0 & j = 0 \\ \mathbb{Z}_2 & 1 \leq j = \text{odd} < n \\ 0 & 0 < j = \text{even} \end{cases}, \qquad \hom(H_i(X), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i = 0 \\ \mathbb{Z}_2 & 1 \leq i = \text{odd} < n \\ 0 & 0 < i = \text{even} \end{cases}$$

$$\Rightarrow H^i(X; \mathbb{Z}_2) \cong \mathrm{Ext}(H_{i-1}(X), \mathbb{Z}_2) \oplus \hom(H_i(X), \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \forall\ 0 \leq i \leq n.$$

<u>Exc.</u> Carry out the calculation of $H^i(\mathbb{R}P^n; \mathbb{Z}_2)$ for $n = \text{odd}$.

Main lemma. 1) Let $F_\bullet \xrightarrow{\varepsilon} H$ be a free resolution of H and $F'_\bullet \xrightarrow{\varepsilon'} H$ a resolut. (not necess. free) of H'. Then every homo. $\alpha: H \to H'$ can be extended to a chain map $F_\bullet \to F'_\bullet$, i.e.,

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "\varepsilon"] \arrow[d, "\alpha_0"] & H \arrow[d, "\alpha"] \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"] & F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "\varepsilon'"] & H'
\end{tikzcd}.$$

Moreover, every two such extensions are chain homotopic.

2) For every two free resolutions $F_\bullet$ & $F'_\bullet$ of H, $\exists$ a canonical iso.

$H^n(F; G) \cong H^n(F'; G)$.

Proof of the lemma. We'll define α_i by induction. Take $\alpha_{-1} := \alpha$.

$f_0 := \varepsilon$

$f_0' := \varepsilon'$

$\alpha_{-1} := \alpha$

Choose a basis $\{x_s\}$ for F_0. $f_0' = \varepsilon'$ is surjective, so $\exists$ $x'_s \in F_0'$ s.t. $f_0'(x'_s) = \alpha f_0(x_s)$. Define $\alpha_0(x_s) := x'_s$. Since F_0 is free this defines α_0 uniquely & $f_0' \alpha_0 = \alpha_{-1} f_0$.

Now suppose we've already defined $\alpha_{-1}, \alpha_0, \dots, \alpha_i$

$$\begin{tikzcd}
F_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, dashed, "\alpha_{i+1}"'] & F_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"] & F_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "\alpha_{i-1}"] & \cdots \arrow[r] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha_{-1}"] & 0 \\
F'_{i+1} \arrow[r, "f'_{i+1}"'] & F'_i \arrow[r, "f'_i"'] & F'_{i-1} \arrow[r, "f'_{i-1}"'] & \cdots \arrow[r] & F'_0 \arrow[r, "f'_0"'] & H' \arrow[r] & 0
\end{tikzcd}$$

choose a basis $\{x_s\}$ for F_{i+1}. $\forall$ basis element x_s, $\alpha_i f_{i+1}(x_s) \in \mathrm{image}(f'_{i+1})$

b.c. $\alpha_i f_{i+1}(x_s) \in \ker(f'_i)$ ($\leftarrow$ this is b.c. $f'_i \alpha_i f_{i+1}(x_s) = \alpha_{i-1} f_i f_{i+1}(x_s) = 0$)

Define $\alpha_{i+1}(x_s) := x'_s$, where $x'_s \in \left(f'_{i+1}\right)^{-1}\left(\alpha_i f_{i+1}(x_s)\right)$.

This proves the existence of α_j $\forall j$.

<u>Uniqueness up to ch. homotopy.</u> If $\{\alpha_i\}$ & $\{\alpha'_i\}$ are two extensions of α, we have to show that $\alpha_i - \alpha'_i$ is ch. homotopic to 0.

Note that $\{\alpha_i - \alpha'_i\}$ is an extension of $0 : H \longrightarrow H'$. So, it's enough to prove that if $\{\beta_i\}$ is an extension of $H \xrightarrow{0} H'$ then $\exists$ a ch. homotopy $h_i : F_i \longrightarrow F'_{i+1}$, $i = -1, 0, 1, 2, \ldots$ s.t. $f'_{i+1} \circ h_i + h_{i-1} \circ f_i = \beta_i$.

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\beta_1"] \arrow[dl, "h_0"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0"] \arrow[dl, "h_{-1}"'] & H \arrow[r] \arrow[d, "0"] \arrow[dl] & 0 \\
\cdots \arrow[r] & F'_1 \arrow[r, "f'_1"'] & F'_0 \arrow[r, "f'_0"'] & H' \arrow[r] & 0
\end{tikzcd}$$

~~Pf~~ Induction on i. For $i=-1$, take $h_{-1}=0$. Then h_0 has to satisfy $\beta_0(x) = f_1' h_0(x)$ $\forall\, x \in F_0$. Again, choose a basis $\{x_s\}$ for F_0.

$\forall$ basis element x_s, $\beta_0(x_s) \in \ker(f_0') = \text{image}(f_1')$. So, $\exists\ x_s' \in F_1'$ s.t. $f_1'(x_s') = \beta_0(x_s)$. Define $h_0(x_s) := x_s'$.

Let $i \geq 1$. Suppose we've already defined $h_{-1}, h_0, \ldots, h_{i-1}$ s.t.

$f_i' h_{i-1} + h_{i-2} f_{i-1} = \beta_{i-1}$. We'll define now h_i, s.t. $f_{i+1}' h_i(x) + h_{i-1} f_i(x) = \beta_i(x)$ $\forall\, x \in F_i$.

$$\begin{tikzcd}
F_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\beta_{i+1}"] & F_i \arrow[r, "f_i"] \arrow[d, "\beta_i"] \arrow[dl, dashed, "h_i"'] & F_{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "\beta_{i-1}"] \arrow[dl, "h_{i-1}"'] & F_{i-2} \arrow[d, "\beta_{i-2}"] \arrow[dl, "h_{i-2}"'] \\
F_{i+1}' \arrow[r, "f_{i+1}'"'] & F_i' \arrow[r, "f_i'"'] & F_{i-1}' \arrow[r, "f_{i-1}'"'] & F_{i-2}'
\end{tikzcd}$$

Choose a basis $\{x_s\}$ of F_i. If we know that $y_s := \beta_i(x_s) - h_{i-1} f_i(x_s) \in \text{image}(f_{i+1}')$ $= \ker(f_i')$

then we are done: just define $h_i(x_s) := x_s'$ for some choice $x_s' \in F_{i+1}'$ s.t. $f_{i+1}'(x_s') = y_s$. Indeed $f_i'(y_s) = f_i'\beta_i(x_s) - f_i' h_{i-1} f_i(x_s) =$

$$\overset{\text{induction}}{=} \beta_{i-1} f_i(x_s) - \left(\beta_{i-1} - h_{i-2} f_{i-1}\right)\cdot f_i(x_s) = \beta_{i-1} f_i(x_s) - \beta_{i-1} f_i(x_s) = 0$$

$\Rightarrow\ y_s \in \ker(f_i')$ as we wished. This completes the induction.

Let G be an ab. group. Let $\{\alpha_i\}, \{\gamma_i\}$ be two extensions of $\alpha: H \to H'$.

Consider $\alpha^*: H'^* \to H^*$ & the coch. maps $\alpha_i^*: F_i'^* \to F_i^*$, $\gamma_i^*: F_i'^* \to F_i^*$

$$\begin{tikzcd} 0 \arrow[r] & H^* \arrow[r] & F_0^* \arrow[r] & \cdots \\ 0 \arrow[r] & H'^* \arrow[r] \arrow[u, "\alpha^*"] & F_0'^* \arrow[r] \arrow[u, "\alpha_0^*"', "\gamma_0^*"] & \cdots \end{tikzcd}$$

Since $\underset{\text{ch. homot.}}{\alpha_0 \simeq \gamma_0} \Rightarrow \underset{\text{coch. homot.}}{\alpha_0^* \simeq \gamma_0^*}$

$\Rightarrow$ the induced maps in cohomology coincide; $\alpha_i^* = \gamma_i^*: H^i(F'; G) \to H^i(F; G)$.

In particular, we get a canonical map $\alpha^{ext}: \mathrm{Ext}(H', G) \longrightarrow \mathrm{Ext}(H, G)$ that depends only on $\alpha: H \to H'$.

Now, let H, H', H'' be ab. groups and $F_\cdot, F_\cdot', F_\cdot''$ resolut. of H, H', H'' respectively with $F_\cdot$ & $F_\cdot'$ being free. Let $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ be homo.

$\Rightarrow (\beta \circ \alpha)_i^* = \alpha_i^* \circ \beta_i^*: H^i(F_\cdot''; G) \longrightarrow H^i(F_\cdot; G)$. (*)

In particular $(\beta\circ\alpha)^{ext} = \alpha^{ext} \circ \beta^{ext}: \mathrm{Ext}(H'', G) \longrightarrow \mathrm{Ext}(H, G)$.

The reason for this (i.e. (*)) is that we can choose the extension of $\beta\circ\alpha$ to be $\beta_i \circ \alpha_i: F_i \to F_i''$ $\forall i$.

Lecture #4B.

<u>Last time:</u> Given $\alpha: H \to H'$, and free resolutions $F_\cdot \xrightarrow{f} H$, $F'_\cdot \xrightarrow{f'} H'$

we obtain an extension $\alpha_\cdot$
$$\begin{tikzcd} F_\cdot \arrow[r, "f"] \arrow[d, dashed, "\alpha_\cdot"'] & H \arrow[d, "\alpha"] \\ F'_\cdot \arrow[r, "f'"] & H' \end{tikzcd}$$
which is unique up to chain homotopy.

$\Rightarrow$ canonical map $H^i(F'_\cdot; G) \xrightarrow{\alpha_\cdot^*} H^i(F_\cdot; G)$. $\Rightarrow$ Canonical homo. $\boxed{\alpha^{ext}: \mathrm{Ext}(H', G) \longrightarrow \mathrm{Ext}(H, G).}$

If $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ we can take $\beta_\cdot \circ \alpha_\cdot$ as an extension of $\beta \circ \alpha$

$$\Longrightarrow \quad \begin{tikzcd} \mathrm{Ext}(H'', G) \arrow[r, "\beta^{ext}"] \arrow[rr, bend right, "(\beta\circ\alpha)^{ext}"'] & \mathrm{Ext}(H', G) \arrow[r, "\alpha^{ext}"] & \mathrm{Ext}(H, G) \end{tikzcd}$$

$$\begin{tikzcd} F_\cdot \arrow[r, "f"] \arrow[d, "\alpha_\cdot"] \arrow[dd, bend right, "\beta_\cdot\circ\alpha_\cdot"'] & H \arrow[d, "\alpha"] \\ F'_\cdot \arrow[r, "f'"] \arrow[d, "\beta_\cdot"] & H' \arrow[d, "\beta"] \\ F''_\cdot \arrow[r, "f''"] & H'' \end{tikzcd}$$

$$\boxed{(\beta\circ\alpha)^{ext} = \alpha^{ext} \circ \beta^{ext}}$$

Consider now two free resolut. $F_\cdot$ & $F'_\cdot$ of the same group H.
We want to show that $\exists$ a canonical iso. $H^i(F_\cdot; G) \cong H^i(F'_\cdot; G)$,
in particular a canonical iso. $H^1(F_\cdot; G) \cong H^1(F'_\cdot; G)$, hence $\mathrm{Ext}(H, G)$
is well defined (i.e. independent of the free resolut. of H, up to can. iso.).

Consider $id: H \to H$. We obtain two possible extensions of this homo.

$\alpha_\cdot : F_\cdot \longrightarrow F'_\cdot$ and $\beta_\cdot : F'_\cdot \longrightarrow F_\cdot$

$$\begin{tikzcd}
F_\cdot \arrow[r, "f"] \arrow[d, dashed, "\alpha"'] \arrow[dd, bend right=60, "id_\cdot"'] & H \arrow[d, "id"] \\
F'_\cdot \arrow[r, "f'"] \arrow[d, dashed, "\beta"'] & H \arrow[d, "id"] \\
F_\cdot \arrow[r, "f"] & H
\end{tikzcd}$$

Now $\beta_\cdot \circ \alpha_\cdot : F_\cdot \longrightarrow F_\cdot$ is an ext. of id.

Also $id_\cdot : F_\cdot \longrightarrow F_\cdot$ ———— " ———— id.

$$\alpha_i^* \circ \beta_i^* = (\beta_i \circ \alpha_i)^* = id^* = id : H^i(F; G) \longrightarrow H^i(F; G).$$

similarly $\beta_i^* \circ \alpha_i^* = (\alpha_i \circ \beta_i)^* = id^* = id : H^i(F'; G) \longrightarrow H^i(F'; G)$

$\Rightarrow$ α_i^* & β_i^* are iso.'s. Moreover α_i^* & β_i^* are canonical.

In particular $(\beta \circ \alpha)^{ext} = id^{ext} = id$

$$\begin{array}{c} (\beta\circ\alpha)^{ext} \\ \shortparallel \\ \alpha^{ext} \circ \beta^{ext} \end{array}$$

∎

UCT for tensor products.

Recall: Let $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be an exact seq. of R-modules.

Then $\forall$ R-mod. M, the seq. $M \otimes_R U \xrightarrow{\mathrm{id}\otimes f} M \otimes_R V \xrightarrow{\mathrm{id}\otimes g} M \otimes_R W \to 0$ is exact.

But $\exists$ SES $0 \to U \to V \to W \to 0$ for which we lose exactness from the left after $M \otimes_R -$.

If M is free $\Rightarrow$ $0 \to M \otimes_R U \to M \otimes_R V \to M \otimes_R W \to 0$ is exact.

Also, if $0 \to U \to V \to W \to 0$ splits $\Rightarrow$ $0 \to M \otimes_R U \to M \otimes_R V \to M \otimes_R W \to 0$ is exact $\forall$ ~~...~~ R-mod. M.

(as a seq. of R-modules)

Let $(C_\cdot, \partial)$ be a ch. complex of free ab. groups.

Here $R = \mathbb{Z}$. $\otimes = \otimes_{\mathbb{Z}}$.

Q. What is the relation between $H_*(C_\cdot \otimes G)$ and $H_*(C_\cdot)$, G ?

diff is $\partial \otimes \mathrm{id}$

denote this also by $H_*(C_\cdot ; G)$.

Like before, consider $B_k \subset Z_k \subset C_k$, $i_k : B_k \longrightarrow Z_k$ the inclusion.

consider:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"'] & C_n \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_{n-1} \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This is a SES ~~sets~~ of chain complexes, where on $B_\cdot$ & $Z_\cdot$ we take the diff. to be 0.

$B_k \subset C_k$ is free b.c. C_k is free $\Longrightarrow$ ~~then~~ every row of the seq. is split when viewed as a seq. of abelian groups. $\Longrightarrow$ after $-\otimes G$ we still obtain SES's:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "0"'] & C_n \otimes G \arrow[r] \arrow[d, "d"'] & B_{n-1} \otimes G \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] \arrow[d] & C_{n-1} \otimes G \arrow[r] \arrow[d] & B_{n-2} \otimes G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$d := \partial \otimes id$

Passing to hlgy: we get a LES:

$$\cdots \longrightarrow B_n \otimes G \xrightarrow{c_n} Z_n \otimes G \longrightarrow H_n(C;G) \longrightarrow B_{n-1} \otimes G \xrightarrow{c_{n-1}} Z_{n-1} \otimes G \to \cdots$$

connect. connect.

<u>claim.</u> $c_n = i_n \otimes id$, $c_{n-1} = i_{n-1} \otimes id$ (exc.)

Now break the LES into many SES's:

$$0 \to \operatorname{coker}(i_n \otimes id) \longrightarrow H_n(C;G) \longrightarrow \ker(i_{n-1} \otimes id) \longrightarrow 0 \qquad (*)$$

Lemma. Let $f: U \to V$ be a homo. of R-modules. Let M be an R-mod. Consider $f \otimes id : U \otimes_R M \to V \otimes_R M$. Then $\exists$ a canonical iso.

$$\operatorname{coker}(f \otimes id) \cong \operatorname{coker}(f) \otimes_R M.$$

Proof. Let N be an R-mod. and $I \subset N$ a submodule.

Consider $0 \to I \xrightarrow{i} N \xrightarrow{p} N/I \to 0$ SES of R-modules.

Let M be an R-mod. $\Rightarrow$ $I \otimes_R M \xrightarrow{i \otimes id} N \otimes_R M \xrightarrow{p \otimes id} (N/I) \otimes_R M \to 0$ is exact.

$\Rightarrow$ $N \otimes M / \operatorname{image}(i \otimes id) \cong (N/I) \otimes_R M$. Apply this to $N = V$, $I = \operatorname{image}(f)$.

So $N/I = \operatorname{coker}(f)$. Also note that $\operatorname{image}(i \otimes id) = (i \otimes id)\big(f(U) \otimes M\big) = \operatorname{image}(f \otimes id)$. $\blacksquare$

Going back to $(*)$ and applying the lemma we have $\operatorname{coker}(i_n \otimes id) \cong \operatorname{coker}(i_n) \otimes G = H_n(C) \otimes G$.

$$\Rightarrow \quad 0 \to H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \ker(i_{n-1} \otimes id) \longrightarrow 0$$

$a = [c]$, $g \in G$: $\quad a \otimes g \longmapsto [c \otimes g] \quad$ (exc.)

Analyzing $\ker(i_k \otimes id)$. Consider free resolutions $F_\bullet \xrightarrow{\varepsilon} H$ of a given abelian group H. Given another ab. grp G, we consider $F_\bullet \otimes G \to 0$ which is a ch. complex $\rightsquigarrow$ $H_i(F_\bullet; G) := H_i(F_\bullet \otimes G)$.

Thm. $\forall$ two free resolutions $F_\bullet \xrightarrow{\varepsilon} H$, $F'_\bullet \xrightarrow{\varepsilon'} H$, $\exists$ a canonical iso,

$$H_n(F_\bullet; G) \cong H_n(F'_\bullet; G).$$

So, $H_*(F; G)$ depends only on H & G up to can. iso.

Exe. 1) $H_0(F; G) \cong H \otimes G$. 2) $H_i(F; G) = 0 \quad \forall\, i \geq 2$.

Define $\mathrm{Tor}(H, G) := H_1(F_\bullet; G)$.

Induced maps. If $\alpha : H \to H'$ is a homo. $\Rightarrow \exists$ canonical map $\alpha_{tor} : \mathrm{Tor}(H, G) \to \mathrm{Tor}(H', G)$.

And $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H'' \rightsquigarrow (\beta \circ \alpha)_{tor} = \beta_{tor} \circ \alpha_{tor}$.

Consider now $H := H_{n-1}(C)$. We have a SES $\underset{\deg 2}{0} \to \underset{\deg 1}{B_{n-1}} \xrightarrow{i_{n-1}} \underset{\deg 0}{Z_{n-1}} \to H_{n-1}(C) \to 0$ which gives a free resolut. of $H_{n-1}(C)$. After $-\otimes G$, and taking H_1 we get $\ker(i_{n-1} \otimes id) = \mathrm{Tor}(H_{n-1}(C), G)$.

Thm. Let $(C_\cdot, \partial)$ be a ch. complex of free ab. groups, and G an ab. group.

Then $\exists$ a SES $\quad 0 \to H_n(C)\otimes G \to H_n(C_\cdot \otimes G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0.$

The seq. is nat. w.r.t. chain maps $C_\cdot \to C'_\cdot$ as wells as w.r.t. homo. $G \to G'$. Moreover, the seq. splits (but not canonically).

Properties of Tor. 1) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

2) If A or B are free then $\mathrm{Tor}(A,B) = 0$.

3) $\mathrm{Tor}\left(\bigoplus_{i\in I} A_i, B\right) \cong \bigoplus_{i\in I} \mathrm{Tor}(A_i, B)$.

4) Let $A_{\text{torsion}} \subset A$ be the torsion subgroup of A. Then $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(A_{\text{torsion}}, B)$.

5) $\mathrm{Tor}(\mathbb{Z}_m, A) \cong \ker\left(A \xrightarrow{\times m} A\right)$.

$1-5 \Rightarrow$ calculation of $\mathrm{Tor}(A,B)$ for all finitely gener. ab. groups A & B.

Example. $\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_k$, where $k = \gcd(n,m)$.

Proof. Consider the free resolut. of $\mathbb{Z}_n$: $\quad 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}_n \to 0$

After $\otimes \mathbb{Z}_m$ we get $\quad 0 \to \mathbb{Z}_m \xrightarrow{\times n} \mathbb{Z}_m \to 0 \quad$ for the non-augment. ch. complex.

$$\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \ker(\mathbb{Z}_m \xrightarrow{\times n} \mathbb{Z}_m) \underset{\text{exc.}}{\cong} \mathbb{Z}_k, \qquad k := \text{g.c.d.}(n, m).$$

Property 5 is proved in a very similar way.

Note that also $\mathbb{Z}_k \cong \mathbb{Z}_n \otimes \mathbb{Z}_m$.

<u>Cor.</u> If A, B are f. generated ab. groups $\Longrightarrow \mathrm{Tor}(A,B) \cong A_{\text{torsion}} \otimes B_{\text{torsion}}$.

$$A \cong A_{\text{free}} \oplus A_{\text{torsion}}$$

$$B \cong B_{\text{free}} \oplus B_{\text{torsion}}$$

$$A_{\text{torsion}} \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{n_i}$$

$$B_{\text{torsion}} \cong \bigoplus_{j=1}^{l} \mathbb{Z}_{m_j}$$

Lecture #5A.

Properties of Tor.

1) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

2) $\mathrm{Tor}\left(\bigoplus_{i\in I} A_i, B\right) \cong \bigoplus_{i\in I} \mathrm{Tor}(A_i,B)$.

3) If A or B are free, then $\mathrm{Tor}(A,B)=0$.

4) If A is finitely generated then $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(A_{torsion}, B)$, where $A_{torsion} \subset A$ is the torsion subgroup of A.

5) $\mathrm{Tor}(\mathbb{Z}_m, A) \cong \ker\left(A \xrightarrow{\times m} A\right)$.

6) $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_k$, where $k = \gcd(m,n)$.

Moreover, the iso.'s 1–6 are canonical.

The iso.'s 1, 2, 4 are natural w.r. to homomorphisms of groups for any of the two factors in $\mathrm{Tor}(-,-)$. The iso. 5 is natural w.r.to homomorphisms of groups for the 2'nd factor i.e. $A' \xrightarrow{\alpha} A''$ gives

$$\begin{tikzcd}
\mathrm{Tor}(\mathbb{Z}_m, A') \arrow[r, "\cong"] \arrow[d, "\alpha_{tor}"'] \arrow[dr, phantom, "\copyright"] & \ker(A' \xrightarrow{\times m} A') \arrow[d] \\
\mathrm{Tor}(\mathbb{Z}_m, A'') \arrow[r, "\cong"] & \ker(A'' \xrightarrow{\times m} A'')
\end{tikzcd}$$

induced by α

Outline of proofs.

3) Assume $A =$ free. We have a very short free resolut. of A: $\cdots \to \underset{\deg 1}{0} \to \underset{\deg 0}{A} \xrightarrow{id} \underset{\deg -1}{A} \to 0$

$\Rightarrow$ $\mathrm{Tor}(A,B) = 0$, $\forall$ B.

Assume $B =$ free. Pick a free ab. group F which surjects onto A

$F \xrightarrow{surj} A$. Let $R < F$ be the ker of that surj.

R is also free ab. $\leadsto$ we get a free resolut. of A: $\cdots \to 0 \to R \to F \to A \to 0$

Since B is free, $-\otimes B$ keeps the seq. exact, so the seq.

$0 \to \underset{\deg 1}{R \otimes B} \longrightarrow \underset{\deg 0}{F \otimes B} \longrightarrow A \otimes B \longrightarrow 0$ is exact. $\Rightarrow$ $\mathrm{Tor}(A,B) = 0$.

5) Take the following free resolut. of $\mathbb{Z}_m$: $\cdots 0 \to \underset{\deg 1}{\mathbb{Z}} \xrightarrow{\times m} \underset{\deg 0}{\mathbb{Z}} \longrightarrow \mathbb{Z}_m \longrightarrow 0$

After $-\otimes A$: $\cdots 0 \to A \xrightarrow{\times m} A \longrightarrow \mathbb{Z}_m \otimes A \longrightarrow 0$

$\Rightarrow$ $\mathrm{Tor}(\mathbb{Z}_m, A) = \ker\left(A \xrightarrow{\times m} A\right)$.

Proof that the SES of homological UCT splits.

$$0 \longrightarrow H_n(C.)\otimes G \xrightarrow{h} H_n(C.\otimes G) \longrightarrow \mathrm{Tor}(H_{n-1}(C.), G) \longrightarrow 0.$$

Consider the seq. $0 \longrightarrow Z_n \xrightarrow{j_n} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$. This seq. splits b.c. B_{n-1} is free (b.c. C_{n-1} is free). $\Rightarrow \exists\ Z_n \xleftarrow{p} C_n$ which is a left inverse of j_n, i.e. $p|_{Z_n} = id$. Compose p with the quot. map $Z_n \longrightarrow H_n(C.)$, and we get $\bar{p} : C_n \longrightarrow H_n(C.)$ and we have $\bar{p}(z) = [z]\ \forall z \in Z_n$.

This map exist $\forall n$, so we can view it as a ch. map $\bar{p} : C. \longrightarrow H_*(C.)$, where $H_*(C.)$ is viewed as a ch. complex with 0-differential (indeed $\bar{p}(\partial c) = [\partial c] = 0$). Tensoring with G we get a ch. map

$\bar{p}\otimes id : C.\otimes G \longrightarrow H_*(C.)\otimes G$. Passing to homology we get

$(\bar{p}\otimes id)_* : H_n(C.\otimes G) \longrightarrow H_n(C.)\otimes G$.

<u>Claim.</u> $(\bar{p}\otimes id)_*$ is a left inverse of $H_n(C.)\otimes G \xrightarrow{h} H_n(C.\otimes G)$.

<u>Proof.</u> Let $a = [c] \in H_n(C.)$, with $c \in Z_n$, and let $g \in G$. $\Rightarrow h(a\otimes g) = [c\otimes g]$.
$\Rightarrow (\bar{p}\otimes id)_*([c\otimes g]) = \bar{p}(c)\otimes g = [c]\otimes g = a\otimes g$. $\Rightarrow (\bar{p}\otimes id)_* \circ h = id$. ∎

Ext & Tor for other rings and modules. Not always true that a submodule of a free R-module is free, for general rings. $\rightsquigarrow$ $\mathrm{Tor}_i^R(H, Q) = H_i(F_\bullet \otimes Q)$

R-modules proj. resolut.

In general $\mathrm{Tor}_i^R(H, Q) \neq 0$ for $i \geq 2$.

If $R = \mathbb{Z}$, then $\mathrm{Tor}_1^{\mathbb{Z}} = \mathrm{Tor}$.

__Def.__ A ring R is called a PID (Principal Ideal Domain) if it is an integral domain (i.e. $\nexists$ zero divisors, and $1 \neq 0$), and every ideal $I \subset R$ is principal (i.e. $\exists$ $r \in R$ s.t. $I = r \cdot R$).

__Exps.__ 1) $R = \mathbb{Z}$. 2) $R =$ field. 3) $R = F[x]$, where F is a field.

__Thm.__ Let R be a PID. ~~then~~ Let M be a free R-mod. Then every submodule of M is also free.

__Cor.__ If R is a PID, $\mathrm{Tor}_i^R = 0$ $\forall$ $i \geq 2$, $\mathrm{Ext}_R^i = 0$ $\forall$ $i \geq 2$.

origin of the name Ext: it classifies extensions.

Let A, B be ab. groups. An extension of A by B is a SES

ξ: $0 \to B \to X \to A \to 0$. Two extensions ξ & $\xi' = (0 \to B \to X' \to A \to 0)$

are called equivalent if if $\exists$ an iso. $X \xrightarrow[\cong]{f} X'$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commut.

An extension is called split if it is equiv. to $0 \to B \to A \oplus B \to A \to 0$.

$$b \mapsto (0, b) \qquad (a, b) \mapsto a$$

<u>Lemma.</u> If $\mathrm{Ext}(A, B) = 0$ then every extension of A by B is split.

<u>Thm.</u> $\exists$ a bijection

$$\left\{ \begin{array}{l} \text{equiv. classes} \\ \text{of extensions of } A \text{ by } B \end{array} \right\} \longleftrightarrow \mathrm{Ext}(A, B)$$

which sends the split extension to $0 \in \mathrm{Ext}(A, B)$.

Back to Topology. X = space, $A \subset X$ subspace. G = group.

We have a SES: $0 \longrightarrow S_\bullet(A) \longrightarrow S_\bullet(X) \longrightarrow S_\bullet(X,A) \longrightarrow 0$

$S_\bullet(X)/S_\bullet(A)$

Claim. $\forall$ deg. k, this seq. splits as a SES of ab. groups.

Proof. $S_k(X,A)$ is free abelian. One can take a basis for this group to be all the chains $\sigma : \Delta^k \longrightarrow X$ s.t. $\sigma(\Delta^k) \not\subset A$, viewed as elements of $S_k(X)/S_k(A)$. (Exc.) ■

Cor. After $\otimes G$, we still have a SES

$$0 \to S_\bullet(A)\otimes G \longrightarrow S_\bullet(X)\otimes G \longrightarrow S_\bullet(X,A)\otimes G \longrightarrow 0$$

In other words, $$S_\bullet(X)\otimes G \Big/ S_\bullet(A)\otimes G \cong \left(S_\bullet(X)/S_\bullet(A)\right)\otimes G = S_\bullet(X,A)\otimes G.$$

Back to cohomology.

Recall $H^i(X;G) := H^i(S^\cdot(X;G)) = H^i(\hom(S_\cdot(X), G), \delta = \partial^*)$.

$H^0(X;G)$. By UCT

$$0 \to \underbrace{\mathrm{Ext}(H_{-1}(X), G)}_{=0} \longrightarrow H^0(X;G) \xrightarrow{\cong} \hom(H_0(X), G) \longrightarrow 0$$

$$\Rightarrow \quad H^0(X;G) \cong \hom\Big(\bigoplus_{c\in\pi_0(X)} \mathbb{Z}, G\Big) = \prod_{c\in\pi_0(X)} G \qquad \left(\begin{array}{l}\text{locally constant}\\ G\text{-valued functions on } X\end{array}\right)$$

$H^1(X;G)$. Since $H_0(X) = $ free we have $\mathrm{Ext}(H_0(X), G) = 0$,

so by UCT $\quad H^1(X;G) \cong \hom(H_1(X), G) \cong \hom(\pi_1(X), G)$

(under the assumpt. that X = path-connected)

b.c. G is abelian and $H_1(X) = \pi_1(X)^{ab}$

Let F = a field. Then

$$H^n(X;F) \cong \hom_F(H_n(X;F), F).$$

Proof requires $\mathrm{Ext}^\cdot_F$.

Reduced cohomology. Assume $X \neq \phi$.

Consider the augmented complex:
$$\cdots \longrightarrow S_1(X) \longrightarrow S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$
$$\text{simplex } \sigma \longmapsto 1$$

To define $\tilde{H}^n(X;G)$ we take $\hom(-,G)$:

$$\tilde{H}^n(X;G) = \begin{cases} H^n(X;G) & n>0 \\ \hom(\tilde{H}_0(X),G) & n=0 \end{cases}$$

exc. follows from UCT, applied to the augmented ch. complex.

Exc. $\tilde{H}^0(X;G) = \dfrac{\{\text{loc. const. funct. } \varphi: X \longrightarrow G\}}{\{\text{(glob.) const. funct. } X \longrightarrow G\}}$.

Relative cohomology. $A \subset X$. $\rightsquigarrow$ $0 \to S_\bullet(A) \xrightarrow{i} S_\bullet(X) \xrightarrow{j} S_\bullet(X,A) \longrightarrow 0$

Recall this seq. splits as seq. of ab. groups, so after $\hom(-,G)$ we still have a SES:
$$0 \to S^\bullet(X,A;G) \longrightarrow S^\bullet(X;G) \xrightarrow{i^*} S^\bullet(A;G) \longrightarrow 0$$

restrict. map.

$S^n(X,A;G) =$ homo. $\varphi : S_n(X) \longrightarrow G$ s.t. $\varphi|_{S_n(A)} \equiv 0$.

In cohomology we get a LES

$$\cdots \xrightarrow{\delta^*} H^n(X,A;G) \longrightarrow H^n(X;G) \longrightarrow H^n(A;G) \xrightarrow{\delta^*} H^{n+1}(X,A;G) \longrightarrow \cdots$$

connect.

<u>Exc.</u>

$$\begin{tikzcd} H^n(A;G) \arrow[r, "\delta^*"] \arrow[d, "h"'] \arrow[dr, phantom, "\copyright"] & H^{n+1}(X,A;G) \arrow[d, "h"] \\ \hom(H_n(A),G) \arrow[r, "c"] & \hom(H_{n+1}(X,A),G) \end{tikzcd}$$

where c is the dual map of $H_{n+1}(X,A) \xrightarrow{\partial_*} H_n(A)$

<u>Induced maps.</u> $f:(X,A) \longrightarrow (Y,B) \rightsquigarrow f_c : S_\cdot(X,A) \longrightarrow S_\cdot(Y,B)$

$\rightsquigarrow$ coch. map $f_c^* : S^\cdot(Y,B;G) \longrightarrow S^\cdot(X,A;G) \rightsquigarrow f^*: H^n(Y,B;G) \longrightarrow H^n(X,A;G)$

The LES of (X,A) & (Y,B) are related by:

$$\begin{tikzcd} \cdots \arrow[r, "\delta^*"] & H^n(Y,B;G) \arrow[r] \arrow[d, "f^*"'] & H^n(Y;G) \arrow[r] \arrow[d, "f^*"'] & H^n(B;G) \arrow[r, "\delta^*"] \arrow[d, "f^*"'] & \cdots \\ \cdots \arrow[r, "\delta^*"] & H^n(X,A;G) \arrow[r] & H^n(X;G) \arrow[r] & H^n(A;G) \arrow[r, "\delta^*"] & \cdots \end{tikzcd}$$

$(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$

$(f\circ g)^* = g^* \circ f^*$ etc.

Cohomology of spaces (continued).

1) $\exists$ a SES $0 \longrightarrow \mathrm{Ext}(H_{n-1}(X,A),G) \longrightarrow H^n(X,A;G) \longrightarrow \hom(H_n(X,A),G) \longrightarrow 0$
coming from UCT. The seq. is nat. w.r. to maps $(X,A) \longrightarrow (Y,B)$.

We can apply UCT to $S.(X,A)$ b.c. $\forall$ k, $S_k(X,A)$ is a free ab. group.

2) $\exists$ also a homological version involving $H_n(X,A)\otimes G$, $\mathrm{Tor}(H_{n-1}(X,A),G)$ & $H_n(X,A;G)$.

The sequences split but not canonically and in fact ~~the~~ a splitting cannot always be arranged to be nat. w.r. to maps between spaces.

Homotopy invariance. If $f,g:(X,A)\longrightarrow (Y,B)$, and $f\simeq g$

$\Longrightarrow f^* = g^* : H^n(Y,B;G) \longrightarrow H^n(X,A;G)$.

Proof. $\exists$ a ch. homotopy between f_c & g_c, i.e. a homo. $H: S_n(X,A)\longrightarrow S_{n+1}(Y,B)$ $\forall n$, s.t. $g_c - f_c = \partial\circ H + H\circ\partial$. $\Rightarrow$ $g_c^* - f_c^* = H^*\circ\partial^* + \partial^*\circ H^*$.
So H^* gives a coch. homotopy between f_c^* & g_c^*.

Excision. $A \subset X$. We have $Z \subset A$, and assume $\overline{Z} \subset \operatorname{Int}(A)$.

$\Rightarrow$ $i : (X \setminus Z, A \setminus Z) \longrightarrow (X, A)$ induces an iso. $H^n(X, A; G) \xrightarrow[\cong]{i^*} H^n(X \setminus Z, A \setminus Z; G)$.

Proof. 1'st proof: dualize the proof for homology.

2'nd proof. Denote by $i_h : H_*(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_*(X, A)$ the map induced by i.

We have:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(X \setminus Z, A \setminus Z), G) \arrow[r] & H^n(X \setminus Z, A \setminus Z; G) \arrow[r] & \hom(H_n(X \setminus Z, A \setminus Z), G) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, "\cong", "||"'] & \operatorname{Ext}(H_{n-1}(X, A), G) \arrow[r] \arrow[u, "i_h^{ext}"', "\cong"] & H^n(X, A; G) \arrow[r] \arrow[u, "i^*"', "\cong"] & \hom(H_n(X, A), G) \arrow[r] \arrow[u, "(i_h)^*"', "\cong"] & 0 \arrow[u, "\cong", "||"']
\end{tikzcd}$$

The statement follows from the 5-lemma. $\blacksquare$

Cellular cohomology. Let X be a CW-complex, G ab. group.

$\rightsquigarrow$ $C_\bullet^{cw}(X)$ cellular ch. complex. $C^\bullet_{cw}(X; G) := \hom(C_\bullet^{cw}(X), G)$.

Thm. $H^*(C^\bullet_{cw}(X); G) \cong H^*(X; G)$. Works also for pairs (X, A).

Proof. We'll apply UCT.

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(H_{n-1}^{cw}(X), G) & \longrightarrow & H^n(C^{\bullet}_{cw}(X;G)) & \longrightarrow & \hom(H_n^{cw}(X), G) & \longrightarrow 0 \\
& \| \wr & & & & \| \wr & \\
0 \to & \mathrm{Ext}(H_{n-1}(X), G) & \longrightarrow & H^n(X;G) & \longrightarrow & \hom(H_n(X), G) & \longrightarrow 0
\end{array}$$

We know that the SES's in UCT split, so

$$\begin{aligned}
H^n(C^{\bullet}_{cw}(X;G)) &\cong \mathrm{Ext}(H_{n-1}^{cw}(X), G) \oplus \hom(H_n^{cw}(X), G) \\
&\cong \mathrm{Ext}(H_{n-1}(X), G) \oplus \hom(H_n(X), G) \cong H^n(X;G).
\end{aligned}$$

$\blacksquare$

The MV LES in cohomology. X space, $A, B \subset X$ subspaces.

Assume: $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. Then $\exists$ a LES

$$\cdots \longrightarrow H^n(X;G) \longrightarrow H^n(A;G) \oplus H^n(B;G) \longrightarrow H^n(A \cap B;G) \longrightarrow H^{n+1}(X;G) \longrightarrow \cdots \qquad (*)$$

This comes from the SES:

$$0 \longrightarrow S_n(A \cap B) \longrightarrow S_n(A) \oplus S_n(B) \longrightarrow S_n^{A,B}(X) \longrightarrow 0$$

(subgroup of $S_n(X)$ generated by the chains that are either in A or in B)

We'd like to dualize, by $\hom(-, G)$ and get a SES of coch. complexes

(possible b.c. $S_n^{A,B}(X) \subset S_n(X)$ is free, b.c. $S_n(X)$ is free, so the seq. splits.)

After dualizing, we use the fact that $S_\cdot^{A,B}(X) \longrightarrow S_\cdot(X)$ induces an iso in homology, and from this we'll obtain that $S^\cdot(X;G) \longrightarrow S^\cdot_{A,B}(X;G) := \hom(S_\cdot^{A,B}(X), G)$ induces an iso. in cohomology. Passing to cohomology we get the seq. $(*)$.

The cochain seq. $0 \longrightarrow S^\cdot_{A,B}(X;G) \longrightarrow S^\cdot(A;G) \oplus S^\cdot(B;G) \longrightarrow S^\cdot(A \cap B;G) \longrightarrow 0$

$$h \longmapsto (h|_{S_\cdot(A)}, h|_{S_\cdot(B)})$$

$$(f, g) \longmapsto f|_{S_\cdot(A \cap B)} - g|_{S_\cdot(A \cap B)}.$$

Another argument why the dual of the seq. $0 \longrightarrow S_n(A\cap B) \longrightarrow S_n(A)\oplus S_n(B) \longrightarrow S_n^{A,B}(X) \longrightarrow 0$

is exact from the right.

$$0 \longrightarrow S^{\bullet}_{A,B}(X;G) \longrightarrow S^{\bullet}(A;G)\oplus S^{\bullet}(B;G) \longrightarrow S^{\bullet}(A\cap B;G) \longrightarrow 0$$

We need to show that if $\varphi : S_n(A\cap B) \longrightarrow G$ is a cochain, then $\exists\ \phi : S_n(A) \longrightarrow G$ and $\Psi : S_n(B) \longrightarrow G$ s.t. $\phi\big|_{S_n(A\cap B)} - \Psi\big|_{S_n(A\cap B)} = \varphi$.

Indeed, we can extend φ to a G-valued funct. on the sing. simplices of A,

$$\phi(\sigma) := \begin{cases} \varphi(\sigma) & \sigma(\Delta)\subset A\cap B \\ 0 & \sigma(\Delta)\not\subset A\cap B. \end{cases} \qquad \text{Take } \Psi := 0.$$

$\sigma : \Delta^n \longrightarrow A$

<u>Motivating question.</u> What is the relation between $H_*(X\times Y)$ & $H_*(X)$, $H_*(Y)$.

<u>Graded groups.</u> $A_\bullet = \{A_i\}_{i\in\mathbb{Z}}$ graded abelian group. Sometimes we write

$$\left(A = \bigoplus_{i\in\mathbb{Z}} A_i .\right)$$

$A_\bullet, B_\bullet$ are graded ab. group. $f: A \to B$ homo.

We say f is graded of deg. d, if $f(A_i) \subset B_{i+d}$ $\forall\, i$. We write $|f| = d$.

<u>Tensor products.</u> $A_\bullet, B_\bullet$ graded ab. groups. $\Rightarrow$ $A\otimes B$ inherits a grading.

$$(A\otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j .$$

Let $f: A'_\bullet \to B'_\bullet$, $g: A''_\bullet \longrightarrow B''_\bullet$ be graded homo.'s. Then $\exists$ a graded homo.

$f\otimes g : A'\otimes A'' \longrightarrow B'\otimes B''$ which satisfies: $(f\otimes g)(a'\otimes a'') = (-1)^{|g|\cdot|a'|} f(a')\otimes g(a'')$

$\forall$ a', a'' elements of pure degree.

$|f\otimes g| = |f| + |g|$.

$$\langle f\otimes g, a'\otimes a''\rangle = (-1)^{|g|\cdot|a'|} \langle f, a'\rangle \otimes \langle g, a''\rangle$$

Chain complexes. $(A_\bullet, \partial_A)$, $(B_\bullet, \partial_B)$ be ch. complexes. $\rightsquigarrow (A \otimes B, \partial_{A\otimes B})$

$$\partial_{A\otimes B} := \partial_A \otimes id_B + id_A \otimes \partial_B. \quad \text{Using our new sign conventions} \begin{pmatrix} |\partial_A| = -1 \\ |\partial_B| = -1 \end{pmatrix}.$$

$$\partial_{A\otimes B}(a\otimes b) = (\partial_A \otimes id_B)(a\otimes b) + (id_A \otimes \partial_B)(a\otimes b) =$$

$$= \partial_A(a)\otimes b + (-1)^{(-1)\cdot|a|} a \otimes \partial_B(b) = \partial_A(a)\otimes b + (-1)^{|a|} a \otimes \partial_B(b).$$

exc. check that $\partial_{A\otimes B} \circ \partial_{A\otimes B} = 0$.

Special case: $(A_\bullet, \partial_A)$ ch. complex
G - ab. group, viewed as a ch. complex concentr-ated in deg. 0.
$\rightsquigarrow A_\bullet \otimes G$ coincides with our previous construct.

Let X, Y be spaces. $\rightsquigarrow S_\bullet(X), S_\bullet(Y), S_\bullet(X\times Y), S_\bullet(X)\otimes S_\bullet(Y)$.

Thm. (Eilenberg-Zilber) $\exists$ a ch. homotopy equiv.

$$S_\bullet(X\times Y) \simeq S_\bullet(X)\otimes S_\bullet(Y)$$

that is natural in X & Y.

In particular, $\exists$ iso. $H_*(X\times Y) \cong H_*(S_\bullet(X)\otimes S_\bullet(Y))$.

Lecture #6A.

The homology cross product.

X = space, denote 0-simplices in X by $x \in X$.

Thm. $\forall$ two spaces X & Y $\exists$ a bilinear map
$$\begin{array}{ccc} S_p(X) \times S_q(Y) & \xrightarrow{\ \times\ } & S_{p+q}(X \times Y) \\ (\sigma, \tau) & \longmapsto & \sigma \times \tau \end{array}$$
$\forall\ p, q \geq 0$, s.t.:

1) $\forall\ x \in X,\ y \in Y,\ \sigma : \Delta^p \longrightarrow X,\ \tau : \Delta^q \longrightarrow Y$

$$x \times \tau : \Delta^q \longrightarrow X \times Y \quad \text{is} \quad (x \times \tau)(v) = (x, \tau(v)) \quad \forall\ v \in \Delta^q$$
$$\sigma \times y : \Delta^p \longrightarrow X \times Y \quad \text{is} \quad (\sigma \times y)(u) = (\sigma(u), y) \quad \forall\ u \in \Delta^p.$$

2) The operation $\times$ is natural in X, Y, namely

if $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ and let $f \times g : X \times Y \longrightarrow X' \times Y'$, $(x, y) \longmapsto (f(x), g(y))$

then $\forall\ a \in S_p(X),\ b \in S_q(Y)$ we have
$$(f \times g)_c\,(a \times b) = f_c(a) \times g_c(b).$$

$$\begin{tikzcd} S_p(X) \times S_q(Y) \arrow[r, "\times"] \arrow[d, "f_c(-) \times g_c(-)"'] & S_{p+q}(X \times Y) \arrow[d, "(f \times g)_c"] \\ S_p(X') \times S_q(Y') \arrow[r, "\times"] & S_{p+q}(X' \times Y') \end{tikzcd}$$

3) $\partial(a \times b) = \partial a \times b + (-1)^{|a|} a \times \partial b \quad \forall\ a \in S_\cdot(X),\ b \in S_\cdot(Y)$ of pure degree.

<u>Rem.</u> Since $\times$ is bilinear, it induces a linear map
$$S_\cdot(X)\otimes S_\cdot(Y) \longrightarrow S_\cdot(X\times Y) \quad \left(\text{in fact } S_p(X)\otimes S_q(Y) \longrightarrow S_{p+q}(X\times Y)\right)$$
$$a\otimes b \longmapsto a\times b$$
We'll denote this operation also by $\times$.

If we endow $S_\cdot(X)\otimes S_\cdot(Y)$ with the diff. $\partial_\otimes := \partial_X \otimes id + id\otimes \partial_Y$, then the map $\times : S_\cdot(X)\otimes S_\cdot(Y) \longrightarrow S_\cdot(X\times Y)$ is a chain map.

Indeed: $(\times \circ \partial_\otimes)(a\otimes b) = \times \cdot \left(\partial a\otimes b + (-1)^{|a|} a\otimes \partial b\right) = \partial a\times b + (-1)^{|a|} a\times \partial b =$
$= \partial(a\times b) = (\partial\circ \times)(a\otimes b).$

<u>Proof of the Thm.</u> Acyclic models.

Induction on $n = p+q$.

<u>$n=0$.</u> So $p=0, q=0$. Define $x\times y := (x,y)$.

For higher n's & the case when $p=0$, or $q=0$, define $x\times\tau$, $\sigma\times y$ as in the statement of the Thm. (exc. check that everything is satisfied).

Let $n\geq 1$, and assume we have already defined $\times$ for all spaces X, Y for all p,q with $0\leq p+q < n$.

Let $0<p$, $0<q$ be s.t. $p+q=n$. Take 1'st $X=\Delta^p$, $Y=\Delta^q$.

Let $i_p : \Delta^p \to \Delta^p$, $i_q : \Delta^q \to \Delta^q$ be the id maps, viewed as sing. simplices.

Consider $a := \partial i_p \times i_q + (-1)^p \, i_p \times \partial i_q \in S_{p+q-1}(\Delta^p \times \Delta^q)$.

(defined already, by induct.)

(This should be $\partial(i_p \times i_q)$ but $i_p \times i_q$ has not yet been defined.)

<u>claim.</u> a is a cycle.

<u>proof.</u> $\partial a = \underbrace{\partial\partial i_p \times i_q}_{=0} + (-1)^{p-1} \partial i_p \times \partial i_q + (-1)^p \partial i_p \times \partial i_q + (-1)^p \cdot (-1)^p \underbrace{i_p \times \partial\partial i_q}_{=0} = 0.$

by the induct. hypothesis. ($|\partial i_p \times i_q| = n-1$, $|i_p \times \partial i_q| = n-1$, so by induct. we can apply the formula for ∂)

But $\Delta^p \times \Delta^q$ is contractible, hence $H_i(\Delta^p \times \Delta^q) = 0 \quad \forall\ i>0$.

Note that $p+q-1>0$, b.c. $p>0$ & $q>0$. $\Rightarrow [a]=0 \in H_{p+q-1}(\Delta^p \times \Delta^q)$

$\Rightarrow \exists\ c \in S_{p+q}(\Delta^p \times \Delta^q)$ s.t. $a = \partial c$. Define $i_p \times i_q := c \in S_{p+q}(\Delta^p \times \Delta^q)$.

Now let $\sigma : \Delta^p \longrightarrow X$, $\tau : \Delta^q \longrightarrow Y$ be sing. simplices.
Note that $\sigma = \sigma_c(i_p)$ & $\tau = \tau_c(i_q)$. Put $\sigma \times \tau := (\sigma \times \tau)_c(i_p \times i_q)$.

<u>Exc.</u> The last def. coincides with the prev. one for the case $X = \Delta^p$, $Y = \Delta^q$, $\sigma = i_p$, $\tau = i_q$.

If $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ are maps, then

$$(f \times g)_c(\sigma \times \tau) = (f \times g)_c (\sigma \times \tau)_c (i_p \times i_q) = \big((f \circ \sigma) \times (g \circ \tau)\big)_c (i_p \times i_q) = f_c(\sigma) \times g_c(\tau),$$

$$\partial(a \times b) = \partial\big((a \times b)_c (i_p \times i_q)\big) = (a \times b)_c \circ \partial(i_p \times i_q) = (a \times b)_c \big(\partial i_p \times i_q + (-1)^p i_p \times \partial i_q\big)$$

assume a, b are sing. simplices.

$$= a_c(\partial i_p) \times b_c(i_q) + (-1)^p a_c(i_p) \times b_c(\partial i_q) = \dots = \partial a \times b + (-1)^p a \times \partial b.$$

$\blacksquare$

A general remark about $C_\bullet \otimes D_\bullet$.

$\exists$ a bilinear map $\quad H_p(C_\bullet)\times H_q(D_\bullet) \xrightarrow{\tilde{h}} H_{p+q}(C_\bullet\otimes D_\bullet)$

that satisfies $\tilde{h}([c],[d]) = [c\otimes d]$, $\forall$ cycles $c\in C_\bullet$, $d\in D_\bullet$.

$\Rightarrow$ $\tilde{h}$ induces a linear map $\quad H_p(C_\bullet)\otimes H_q(D_\bullet) \xrightarrow{h} H_{p+q}(C_\bullet\otimes D_\bullet)$

s.t. $h([c]\otimes[d]) = [c\otimes d]$.

Proof. Exc. Here is an outline: In order to show $\tilde{h}$ is well defined,

we need to check $\big[(c+\partial x)\otimes(d+\partial y)\big] = [c\otimes d]$ $\forall$ cycles c, d, $\forall$ chains x, y.

$$(c+\partial x)\otimes(d+\partial y) = c\otimes d + c\otimes\partial y + \partial x\otimes d + \partial x\otimes\partial y =$$

$$= c\otimes d + \partial(c\otimes y)\cdot(-1)^{|c|} + \partial(x\otimes d) + \partial(x\otimes\partial y) = c\otimes d + \text{boundaries}.$$

$\blacksquare$

Back to top. $C_\bullet = S_\bullet(X)$, $D_\bullet = S_\bullet(Y)$. The above gives us

$$H_p(X)\otimes H_q(Y) \xrightarrow{h} H_{p+q}(S_\bullet(X)\otimes S_\bullet(Y)) \xrightarrow{\times} H_{p+q}(X\times Y)$$

We've seen $\times$ (on ch. level) is a ch. map.

The composition $\times\circ h$ is also denoted sometimes $\times$.

Exc. Let $A \subset X$, $B \subset Y$. Show that $\times$ induces:

$$H_p(X,A) \otimes H_q(Y,B) \xrightarrow{\times} H_{p+q}(X\times Y,\ X\times B \cup A\times Y)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = H_{p+q}\big((X,A)\times(Y,B)\big)$$

Notation:
$(X,A)\times(Y,B) :=$
$= (X\times Y,\ X\times B \cup A\times Y).$

Sign conventions: Koszul sign conventions.

$A_\bullet, B_\bullet$ graded groups. $f: A_\bullet \to B_\bullet$ of deg. d (i.e. $f(A_i) \subset B_{i+d}$ $\forall\, i$). $|f| := d$.

$$A_\bullet \xrightarrow{f} B_\bullet,\quad A'_\bullet \xrightarrow{g} B'_\bullet \quad \rightsquigarrow \quad f\otimes g : A_\bullet \otimes A'_\bullet \longrightarrow B_\bullet \otimes B'_\bullet$$

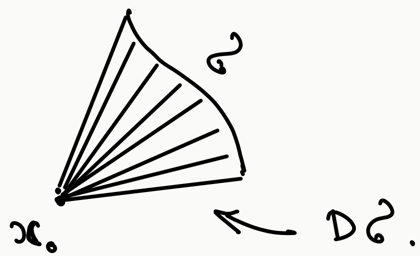
$$\langle f\otimes g,\ a\otimes b\rangle = (-1)^{|g|\cdot|a|} f(a)\otimes g(b). \qquad |f\otimes g| = |f| + |g|.$$

Exc. Let $A_\bullet \xrightarrow{f'} A'_\bullet$, $A'_\bullet \xrightarrow{f''} A''_\bullet$, $B_\bullet \xrightarrow{g'} B'_\bullet$, $B'_\bullet \xrightarrow{g''} B''_\bullet$ be graded homo.
Consider the homo. $(f''\circ f')\otimes(g''\circ g')$ & $(f''\otimes g'')\circ(f'\otimes g')$: $A\otimes B \longrightarrow A''\otimes B''$.
Show that $(f''\circ f')\otimes(g''\circ g') = (-1)^{|f'|\cdot|g''|}(f''\otimes g'')\circ(f'\otimes g')$.

Reminder. 1) X = path connected, $x_0 \in X$. Recall the augment. at x_0. This is the map $\varepsilon_{x_0} : S_\cdot(X) \longrightarrow S_\cdot(X)$, defined by $\varepsilon_{x_0} \equiv 0$ on $S_i(X)$ $\forall i > 0$ and $\varepsilon_{x_0}\left(\sum_{x \in X} n_x x\right) = \left(\sum_{x \in X} n_x\right) \cdot x_0$. Recall that ε_{x_0} is a chain map.

2) Assume X = contractible. Fix $x_0 \in X$. Then $\exists$ a ch. homotopy $D = D_{X, x_0} : S_\cdot(X) \longrightarrow S_\cdot(X)[1]$ s.t. $D\partial + \partial D = id - \varepsilon_{x_0}$. (In particular $H_i(X) = 0$ $\forall i > 0$.)

D was constructed using a cone-construct. and a homotopy $H : X \times [0,1] \longrightarrow X$ with $H(x,0) = x$, $H(x,1) = x_0$ $\forall x \in X$.



Notation. for a chain complex $C_\cdot$ denote by $C[d]$ the ch. complex with the same diff. but with a shift in degree
$$(C[d])_i := C_{i+d}.$$
If $D^\cdot$ is cohomologi-cally graded $(D[d])^i = D^{i-d}$.

Thm. $\exists$ ~~a~~ ch. maps $\Theta : S_\cdot(X \times Y) \longrightarrow S_\cdot(X) \otimes S_\cdot(Y)$, defined $\forall$ spaces X, Y, which is natural in X & Y and s.t. in deg. 0 we have: $\forall\, x \in X,\ y \in Y,\ \Theta((x,y)) = x \otimes y$.

Naturality means: $\forall$ maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ we have a comm. diag.:

$$\begin{tikzcd} S_\cdot(X \times Y) \arrow[r, "\Theta"] \arrow[d, "(f \times g)_c"'] \arrow[dr, phantom, "\textcircled{c}"] & S_\cdot(X) \otimes S_\cdot(Y) \arrow[d, "f_c \otimes g_c"] \\ S_\cdot(X' \times Y') \arrow[r, "\Theta"] & S_\cdot(X') \otimes S_\cdot(Y') \end{tikzcd}$$

Thm. $\exists$ ch. maps $\Theta : S.(X \times Y) \longrightarrow S.(X) \otimes S.(Y)$, defined $\forall$ spaces X, Y, which is natural in X & Y and s.t. in deg. 0 we have:

$\forall\ x \in X,\ y \in Y,\quad \Theta((x,y)) = x \otimes y.$

Naturality means: $\forall$ maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$ we have a comm. diag.:

$$\begin{tikzcd}
S.(X \times Y) \arrow[r, "\Theta"] \arrow[d, "(f \times g)_c"'] \arrow[dr, phantom, "c"] & S.(X) \otimes S.(Y) \arrow[d, "f_c \otimes g_c"] \\
S.(X' \times Y') \arrow[r, "\Theta"] & S.(X') \otimes S.(Y')
\end{tikzcd}$$

We have two functors Pairs of spaces $\longrightarrow$ ch. complexes.

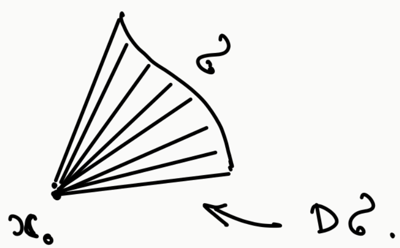
$$(X, Y) \longmapsto S.(X \times Y)$$

$$(X, Y) \longmapsto S.(X) \otimes S.(Y)$$

Reminder. 1) X = path connected, $x_0 \in X$. Recall the augment. at x_0. This is the map $\varepsilon_{x_0} : S_\cdot(X) \longrightarrow S_\cdot(X)$, defined by $\varepsilon_{x_0} \equiv 0$ on $S_i(X)$ $\forall i > 0$ and $\varepsilon_{x_0}\left(\sum_{x\in X} n_x x\right) = \left(\sum_{x \in X} n_x\right)\cdot x_0$. Recall that ε_{x_0} is a chain map.

2) Assume X = contractible. Fix $x_0 \in X$. Then $\exists$ a ch. homotopy $D = D_{X,x_0} : S_\cdot(X) \longrightarrow S_\cdot(X)[1]$ s.t. $D\partial + \partial D = id - \varepsilon_{x_0}$. (In particular $H_i(X) = 0$ $\forall i > 0$.)

D was constructed using a cone-construct. and a homotopy $H : X \times [0,1] \longrightarrow X$ with $H(x,0) = x$, $H(x,1) = x_0$ $\forall x \in X$.



Notation. For a chain complex $C_\cdot$ denote by $C[d]$ the ch. complex with the same diff. but with a shift in degree $(C[d])_i := C_{i+d}$. If $D^\cdot$ is cohomologically graded $(D[d])^i = D^{i-d}$.

<u>Lemma.</u> Let X, Y be contractible spaces, $x_0 \in X$, $y_0 \in Y$. Then $\exists$ a ch. homotopy $E: S(X) \otimes S(Y) \longrightarrow (S(X) \otimes S(Y))[1]$ between $\varepsilon_{x_0} \otimes \varepsilon_{y_0}$ and $id \otimes id$. In particular $H_n(S(X) \otimes S(Y)) = 0 \ \forall \ n \geq 1$, and $\forall$ 0-chain $\sum n_{x,y} \, x \otimes y$ we have $\sum n_{x,y} [x \otimes y] = \left(\sum n_{x,y}\right) [x_0 \otimes y_0]$.

<u>Proof.</u> We'll use the ch. homotopies D_X & D_Y between $id_{S(X)}$ & ε_{x_0} and $id_{S(Y)}$ & ε_{y_0} coming from the fact that X & Y are contractible.

$E := D_X \otimes id + \varepsilon_{x_0} \otimes D_Y$. Recall the diff. d on $S_\cdot(X) \otimes S_\cdot(Y)$

$d = \partial_X \otimes id + id \otimes \partial_Y$. (we use the Koszul sign conventions.)

$$
\begin{aligned}
Ed + dE &= (D_X \otimes id + \varepsilon_{x_0} \otimes D_Y) \circ (\partial_X \otimes id + id \otimes \partial_Y) + \\
&+ (\partial_X \otimes id + id \otimes \partial_Y) \circ (D_X \otimes id + \varepsilon_{x_0} \otimes D_Y) = \\
&= (D_X \circ \partial_X) \otimes id + D_X \otimes \partial_Y - (\varepsilon_{x_0} \circ \partial_X) \otimes D_Y + \varepsilon_{x_0} \otimes (D_Y \circ \partial_Y) + \\
&+ (\partial_X \circ D_X) \otimes id + (\partial_X \circ \varepsilon_{x_0}) \otimes D_Y - D_X \otimes \partial_Y + \varepsilon_{x_0} \otimes (\partial_Y \circ D_Y) = \ldots = id \otimes id - \varepsilon_{x_0} \otimes \varepsilon_{y_0}.
\end{aligned}
$$

▨

<u>Proof of the Thm.</u> Acyclic models. We'll do induction on the degree.

<u>$n=0$.</u> Define $\Theta(x,y) = x \otimes y$.

Let $n \geq 1$, and suppose that Θ has already been defined on $S_k(X \times Y)$ $\forall$ $0 \leq k < n$ and $\forall$ spaces X, Y.

<u>Consider now $k=n$.</u> We 1'st define Θ for the case $X = Y = \Delta^n$ and a very specific chain d_n, where $d_n : \Delta^n \longrightarrow \Delta^n \times \Delta^n$ is the diagonal map ($d_n(x) = (x,x)$). $d_n \in S_n(\Delta^n \times \Delta^n)$.

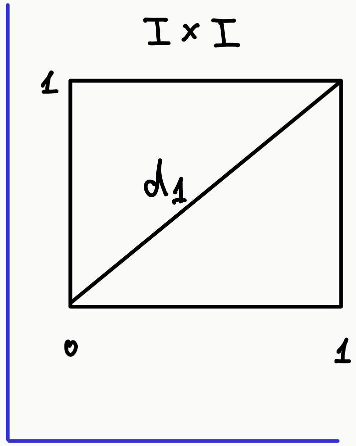
Consider $\partial d_n \in S_{n-1}(\Delta^n \times \Delta^n)$. By induction $\Theta(\partial d_n)$ is already defined.

> Notation: all differentials for all ch. complexes in this proof will be denoted ∂.

<u>claim</u> $\exists$ $a_n \in \left(S_\cdot(\Delta^n) \otimes S_\cdot(\Delta^n)\right)_n$ s.t. $\Theta(\partial d_n) = \partial a_n$.

<u>Proof.</u> $\partial \Theta(\partial d_n) = \Theta(\partial \partial d_n) = 0 \Rightarrow \Theta(\partial d_n)$ is a cycle of deg. $n-1$ in $S_\cdot(\Delta^n) \otimes S_\cdot(\Delta^n)$. If $n \geq 2$ then as Δ^n is contractible we have $H_{n-1}(S(\Delta^n) \otimes S(\Delta^n)) = 0$, hence $\exists$ $a_n \in \left(S(\Delta^n) \otimes S(\Delta^n)\right)_n$ s.t. $\Theta(\partial d_n) = \partial a_n$.
This proves the claim for $n \geq 2$.

If $n=1$, then note that $\Theta(\partial d_1) = \Theta\big((1,1)-(0,0)\big) = 1\otimes 1 - 0\otimes 0$,



and again by the prev. lemma $[\Theta(\partial d_1)] = [1\otimes 1 - 0\otimes 0] = 0$.

So again $\exists\ a_1 \in \big(S(\Delta^1)\otimes S(\Delta^1)\big)_1$ s.t. $\partial a_1 = \Theta(\partial d_1)$.

This proves the claim.

Define $\Theta(d_n) = a_n$. By construction $\partial\Theta(d_n) = \partial a_n = \Theta(\partial d_n)$.

Let now X, Y be spaces and $\sigma : \Delta^n \longrightarrow X\times Y$ be an n-sing. simplex.

Consider $(\pi_X\circ\sigma)\times(\pi_Y\circ\sigma) : \Delta^n\times\Delta^n \longrightarrow X\times Y$.

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Projections.

clearly $\sigma \underset{\substack{\uparrow \\ \text{as maps}}}{=} \big((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma)\big)\circ d_n : \Delta^n \longrightarrow X\times Y$

And $\sigma \underset{\substack{\uparrow \\ \text{as chains.}}}{=} \big((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma)\big)_c(d_n)$

Define $\Theta(\sigma) := \big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c\big)\big(\Theta(d_n)\big)$.

Note that this is the only way to define Θ (once $\Theta(d_n)$ has already been defined) b.c.

$$\Theta(\sigma) := \Big((\pi_X \circ \sigma)_c \otimes (\pi_Y \circ \sigma)_c \Big) \big(\Theta(d_n) \big).$$

$$\begin{tikzcd}
S_n(\Delta^n \times \Delta^n) \arrow[r, "((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma))_c"] \arrow[d, "\Theta"'] & S_n(X\times Y) \arrow[d, "\Theta"] \\
(S(\Delta^n)\otimes S(\Delta^n))_n \arrow[r, "(\pi_X\circ\sigma)_c\otimes(\pi_Y\circ\sigma)_c"] & (S(X)\otimes S(Y))_n
\end{tikzcd}$$

__Exc.__ Check that in case $X=\Delta^n$, $Y=\Delta^n$, $\sigma = d_n$, the new def. coincides with the prev. one.

__Exc.__ Check that Θ as defined above satisfies the naturality condition for maps $X \to X'$, $Y \to Y'$ in degrees $\leq n$.

Let's check also that Θ is a ch. map in deg. $\leq n$.

We need to show $\partial \Theta(\sigma) = \Theta(\partial\sigma)$ $\forall$ $\sigma \in S_n(X\times Y)$.

$\partial\Theta(d_n) = \Theta(\partial d_n)$

$$\partial\Theta(\sigma) = \partial\Big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c \Big)\big(\Theta(d_n)\big) = \Big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c \Big)\, \partial\Theta(d_n) =$$
$$= \Big((\pi_X\circ\sigma)_c \otimes (\pi_Y\circ\sigma)_c \Big)\big(\Theta(\partial d_n)\big) = \Theta\circ\big((\pi_X\circ\sigma)\times(\pi_Y\circ\sigma)\big)_c(\partial d_n) = \Theta\circ\partial\big((\pi_X\circ\sigma)_c\times(\pi_Y\circ\sigma)_c\big)(d_n) =$$
$$= \Theta\, \partial(\sigma).$$

This shows $\circledast$ is a ch. map in deg. $\leq n$, hence completes the induction and the proof of the Thm. $\blacksquare$

<u>Thm.</u> Let ϕ, Ψ be two chain maps, either $S(X\times Y) \longrightarrow S(X\times Y)$, or $S(X)\otimes S(Y) \longrightarrow S(X\times Y)$ or $S(X\times Y) \longrightarrow S(X)\otimes S(Y)$ or $S(X)\otimes S(Y) \longrightarrow S(X)\otimes S(Y)$, defined $\forall$ spaces X, Y and s.t. ϕ & Ψ are natural w.r. to maps between spaces and s.t. ϕ & Ψ are the canonical maps in degree 0. Then $\exists$ a chain homotopy $D_{X,Y}$ between ϕ & Ψ. Moreover we can make the ch. homotopy $D_{X,Y}$ to be natural w.r. to map between spaces $X \to X'$, $Y \to Y'$.

We'll prove here the version for $\phi, \Psi : S(X\times Y) \longrightarrow S(X)\otimes S(Y)$.
We'll define $D : S(X\times Y) \longrightarrow \big(S(X)\otimes S(Y)\big)[1]$ s.t. $D\circ\partial + \partial\circ D = \phi - \Psi$.

Proof. Induction on degree.

$n=0$. Put $D \equiv 0$. This works b.c. $\phi \equiv \Psi$ in degree 0.

Let $n \geq 1$. Assume D has already been defined with all the above properties $\forall\, x, y$ and $0 \leq k < n$. We'll define now D on $S_n(X \times Y)$.

Consider $d_n : \Delta^n \longrightarrow \Delta^n \times \Delta^n$ the diag. map, viewed as an n-simplex in $S_n(\Delta^n \times \Delta^n)$.

$$\partial(\phi - \Psi - D \circ \partial)(d_n) = \partial \phi(d_n) - \partial \Psi(d_n) - \partial D \partial(d_n) =$$

$$\underset{\substack{\text{induction} \\ |\partial(d_n)| = n-1}}{=} \phi(\partial d_n) - \Psi(\partial d_n) - \Big(\underbrace{\phi(\partial d_n)}_{\overset{\shortparallel}{\partial \phi(d_n)}} - \underbrace{\Psi(\partial d_n)}_{\overset{\shortparallel}{\partial \Psi(d_n)}} - D \circ \underbrace{\partial(\partial d_n)}_{\overset{\shortparallel}{0}}\Big) = 0.$$

$\Rightarrow (\phi - \Psi - D \circ \partial)(d_n) \in \big(S(\Delta^n) \otimes S(\Delta^n)\big)_n$ is a cycle. By the lemma from the begin. of the lecture $\exists\, a \in \big(S(\Delta^n) \otimes S(\Delta^n)\big)_{n+1}$ s.t.

$$\partial a = (\phi - \Psi - D \circ \partial)(d_n). \quad \text{Define } D(d_n) := a.$$

Clearly, now we have $(\partial D + D \partial)(d_n) = (\phi - \Psi)(d_n)$.

Now, let $\sigma : \Delta^n \longrightarrow X \times Y$ be a sing. n-simplex. We have

$$\sigma = \big((\pi_X \circ \sigma) \times (\pi_Y \circ \sigma)\big)_c \, (d_n). \quad \text{Define } D\sigma := \big((\pi_X \circ \sigma)_c \otimes (\pi_Y \circ \sigma)_c\big)\big(D(d_n)\big).$$

Exc. complete the proof. $\square$

Corollary (Eilenberg - Zilber Thm.) "The" chain map $\times : S(X) \otimes S(Y) \to S(X \times Y)$ and $\Theta : S(X \times Y) \longrightarrow S(X) \otimes S(Y)$ are uniquely defined up to chain homotopy by their values in deg 0 and the requirement that they are natural in X, Y. Moreover, $\Theta \circ \times \simeq id$, $\times \circ \Theta \simeq id$ via chain homotopies that are nat. in X, Y. In particular $\times$ & Θ are ch. homotopy equivalences, and $\exists$ a natural iso. (w.r.to X, Y)

$$H_*(X \times Y) \cong H_*\big(S_\cdot(X) \otimes S_\cdot(Y)\big).$$

If G is an abelian group then
$H_*(X \times Y; G) \cong H_*\big(S(X) \otimes S(Y) \otimes G\big)$ and $H^*(X \times Y; G) \cong H^*\big(\hom(S(X) \otimes S(Y), G)\big)$.

Lecture #78.

Corollary (Eilenberg - Zilber Thm.) "The" chain map $\times : S(X)\otimes S(Y)\to S(X\times Y)$ and $\Theta : S(X\times Y) \longrightarrow S(X)\otimes S(Y)$ are uniquely defined up to chain homotopy by their values in deg 0 and the requirement that they are natural in X, Y.
Moreover, $\Theta \circ \times \simeq id$, $\times\circ\Theta \simeq id$ via chain homotopies that are nat. in X, Y. In particular $\times$ & Θ are ch. homotopy equivalences, and $\exists$ natural iso. (w.r.to X, Y)

$$H_*(X\times Y) \xrightarrow[\cong]{\Theta_*} H_*(S.(X)\otimes S.(Y))$$

$$H_*(S.(X)\otimes S.(Y)) \xrightarrow[\cong]{\times} H_*(X\times Y)$$

$\Theta_* \circ \times = id$,

$\times \circ \Theta_* = id$.

If G is an abelian group then
$H_*(X\times Y; G) \cong H_*(S(X)\otimes S(Y)\otimes G)$ and $H^*(X\times Y; G) \cong H^*(\hom(S(X)\otimes S(Y), G))$.

Proof follows immediately from:

Thm. Let ϕ, Ψ be two chain maps, either $S(X\times Y)\longrightarrow S(X\times Y)$, or $S(X)\otimes S(Y) \longrightarrow S(X\times Y)$ or $S(X\times Y)\longrightarrow S(X)\otimes S(Y)$ or $S(X)\otimes S(Y)\longrightarrow S(X)\otimes S(Y)$, defined $\forall$ spaces X, Y and s.t. ϕ & Ψ are natural w.r. to maps between spaces and s.t. ϕ & Ψ are the canonical maps in degree 0. Then $\exists$ a chain homotopy $D_{X,Y}$ between ϕ & Ψ. Moreover we can make the ch. homotopy $D_{X,Y}$ to be natural w.r. to map between spaces $X\to X'$, $Y\to Y'$.

The algebraic Künneth formula.

Thm. Let $K_\cdot$ & $L_\cdot$ be ch. complexes of free abelian groups.

Then $\exists$ an exact seq. $\forall n$:

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(K) \otimes H_q(L) \xrightarrow{h} H_n(K \otimes L) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(K), H_q(L)) \longrightarrow 0$$

The 1'st map h has the property $h([k] \otimes [l]) = [k \otimes l]$ $\forall$ cycles k, l.

This SES is natural w.r.t. ch. maps $K_\cdot \longrightarrow K'_\cdot$, $L_\cdot \longrightarrow L'_\cdot$.

The seq. splits but not canonically.

(Exc. check that the above generalizes the UCT)

Topological version. $\forall$ top. spaces X, Y, $\exists$ a SES, $\forall n$

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{\times} H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(K), H_q(L)) \longrightarrow 0$$

The seq. is nat. w.r.t. maps $X \longrightarrow X'$, $Y \longrightarrow Y'$. It splits, but not canonically.
The 1'st map is induced by the cross product.

What happens if we work over a field?

Let $K_\bullet, L_\bullet$ be ch. complexes of vector spaces over a field $\mathbb{F}$.

Then: $\bigoplus_{p+q=n} H_p(K) \otimes_{\mathbb{F}} H_q(L) \xrightarrow{\cong} H_n(K \otimes_{\mathbb{F}} L) \quad \forall\, n.$

Example:
$S_\bullet(X; \mathbb{F}) =$
$= S_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{F}$

__Proof.__ Consider the seq. $0 \to Z_n \xrightarrow{i} K_n \xrightarrow{\partial} B_{n-1} \to 0.$

This is a seq. of v.spaces (over $\mathbb{F}$), hence it splits.

We do now $\otimes$ with $L_\bullet$: we get a SES of ch. complexes

$$0 \to Z \otimes_{\mathbb{F}} L \xrightarrow{i \otimes id} K \otimes_{\mathbb{F}} L \xrightarrow{\partial \otimes id} (B \otimes_{\mathbb{F}} L)[-1] \to 0$$

where Z & B are viewed as ch. complexes with 0-diff.

In homology we get a LES:

$$\cdots \xrightarrow{j \otimes id} (Z \otimes_{\mathbb{F}} H_*(L))_n \longrightarrow H_n(K \otimes_{\mathbb{F}} L) \longrightarrow (B_\bullet \otimes_{\mathbb{F}} H_\bullet(L))_{n-1} \xrightarrow{j \otimes id} (Z_\bullet \otimes_{\mathbb{F}} H_\bullet(L))_{n-1} \longrightarrow \cdots$$

Denote by $j : B_\bullet \hookrightarrow Z_\bullet$ the inclusion. __Exc.__ the connect. homo. is just $j \otimes id$

__Claim.__ $j \otimes id$ is injective. __Proof.__ $H_*(L)$ is a vector space.

$\Rightarrow$ The LES is chopped intwo many SES's:

$$0 \longrightarrow (B_\cdot \otimes_F H_*(L))_n \xrightarrow{j\otimes id} (Z_\cdot \otimes_F H_\cdot(L))_n \longrightarrow H_n(K\otimes L) \longrightarrow 0$$

$$\Rightarrow \quad H_n(K\otimes L) \cong \left(Z_\cdot/B_\cdot \otimes H_\cdot(L)\right)_n = \left(H_\cdot(K)\otimes H_\cdot(L)\right)_n \qquad \blacksquare$$

<u>Back to cohomology.</u>

Let $(A_\cdot, \partial_A)$, $(B_\cdot, \partial_B)$ be two ch. complexes. Define anew cochain complex $\mathrm{hom}(A,B)^\cdot$: $\mathrm{hom}(A,B)^p := \{$graded homo's $A\to B$ of degree $-p\} :=$

$$= \bigoplus_{i\in\mathbb{Z}} \mathrm{hom}(A_i, B_{i-p}).$$

We define $\tilde{\delta}: \mathrm{hom}(A,B)^p \longrightarrow \mathrm{hom}(A,B)^{p+1}$ by the following formula:

$\partial_B\langle f, a\rangle = \langle \tilde{\delta} f, a\rangle + (-1)^p \langle f, \partial_A a\rangle$. This defines $\tilde{\delta}$.

<u>Exc.</u> Show $\tilde{\delta}\circ\tilde{\delta} = 0$. <u>Exc.</u> characterize $f\in \mathrm{hom}(A,B)^0$ that are cocycles.

Special case. $(A_\cdot, \partial)$ ch. complex, G = ab. group. Take $B_\cdot$ to be the ch. complex with $B_i = 0$ $\forall\, i \neq 0$, $B_0 = G$. $(\partial_B = 0)$.

If $f \in \hom(A, G)^p = \hom(A_p, G) \Rightarrow \tilde{\delta} f = (-1)^{p+1} f \circ \partial = (-1)^{|f|+1} \delta f$

clearly $H^i(A^*, \tilde{\delta}) = H^i(A^*, \delta)$.

From now on we'll use only $\tilde{\delta}$ (also for A^*), and denote it from now on by δ.

Cohomological cross product.

Fix a commutative ring R (with a unity).

Let X, Y be spaces. Let $\varphi \in S^p(X;R)$, $\psi \in S^q(Y;R)$ be cochains.

We'll define $\varphi \times \psi \in S^n(X \times Y; R)$, where $n = p+q$.

Write $\varphi : S_p(X) \longrightarrow R$, $\psi : S_q(Y) \longrightarrow R$.

Recall $\Theta : S_\bullet(X \times Y) \longrightarrow S_\bullet(X) \otimes_{\mathbb{Z}} S_\bullet(Y)$. Fix one such map.

Consider $\varphi \otimes \psi : S_p(X) \otimes_{\mathbb{Z}} S_q(Y) \longrightarrow R \otimes_{\mathbb{Z}} R \longrightarrow R$, where the last map is induced by the bilinear map $R \times R \longrightarrow R$, $(r_1, r_2) \longmapsto r_1 \cdot r_2$.

$\Rightarrow$ we get a cochain $\varphi \otimes \psi : \underbrace{\left(S_\bullet(X) \otimes_{\mathbb{Z}} S_\bullet(Y)\right)_{p+q}}_{\left(\bigoplus_{p'+q'=p+q} S_{p'}(X) \otimes_{\mathbb{Z}} S_{q'}(Y)\right)} \longrightarrow R$

by defining $\varphi \otimes \psi$ to be 0

$\forall\ p', q'$ s.t. $p'+q' = p+q$

but $(p', q') \neq (p, q)$.

Define $\varphi\times\psi := (\varphi\otimes\psi)\circ \Theta$ $\qquad \Theta: S_\bullet(X\times Y) \longrightarrow S_\bullet(X)\otimes_{\mathbb{Z}} S_\bullet(Y)$

$\varphi\times\psi \in S^{p+q}(X\times Y;R)$.

Note: $\times$ is natural w.r.t. maps $X\to X'$, $Y\to Y'$, b.c. Θ has this property.

A more explicit formula. Let $c\in S_n(X\times Y)$, where $n=p+q$.

$$\Theta(c) = \sum_{r+s=n}\sum_{i,j} a_i^r\otimes b_j^s, \text{ with } a_i^r\in S_r(X),\ b_j^s\in S_s(Y).$$

$$(\varphi\times\psi)(c) = (-1)^{p\cdot q}\sum_{i,j}\varphi(a_i^p)\cdot\psi(b_j^q).$$ (Koszul sign conventions!)

Claim. $\delta(\varphi\times\psi) = \delta\varphi\times\psi + (-1)^{|\varphi|}\varphi\times\delta\psi$. In other words, the map $S^p(X;R)\otimes_{\mathbb{Z}} S^q(Y;R) \longrightarrow S^{p+q}(X\times Y;R)$ induced by $(\varphi,\psi)\longmapsto \varphi\times\psi$ is a chain map. (w.r.t. the diff. $\delta_X\otimes id + id\otimes\delta_Y$ and $\delta_{X\times Y}$).

Proof of the claim. $p := |\varphi|$, $q := |\psi|$.

$$\delta(\varphi \times \psi) = (-1)^{p+q+1}(\varphi\times\psi)\circ\partial = (-1)^{p+q+1}(\varphi\otimes\psi)\circ\Theta\circ\partial = (-1)^{p+q+1}(\varphi\otimes\psi)\,\underbrace{\partial}_{(\partial_x\otimes id + id\otimes\partial_y)}\circ\,\Theta =$$

$$= (-1)^{p+q+1}\left(\varphi\otimes(\psi\circ\partial_y) + (-1)^q(\varphi\circ\partial_x)\otimes\psi\right)\circ\Theta =$$

$$= (-1)^{p+q+1}\left((-1)^{q+1}\varphi\otimes\delta\psi + (-1)^{q+p+1}\delta\varphi\otimes\psi\right)\circ\Theta =$$

$$= \left(\delta\varphi\otimes\psi + (-1)^p\varphi\otimes\delta\psi\right)\circ\Theta = \delta\varphi\times\psi + (-1)^p\varphi\times\delta\psi. \qquad \blacksquare$$

Remark. The map $(\varphi,\psi)\longmapsto\varphi\times\psi$ is bilinear over R, so it induces a map of R-modules $S^p(X;R)\otimes_R S^q(Y;R)\longrightarrow S^{p+q}(X\times Y;R)$.

Cor. The chain level $\times$ product induces a product

$$H^p(X;R)\otimes_R H^q(Y;R)\longrightarrow H^{p+q}(X\times Y;R)$$

which is independent of the particular choice of Θ.

Lecture #8A.

Let $(A_\cdot, \partial_A)$, $(B_\cdot, \partial_B)$ be two ch. complexes. Define a new cochain complex

$$\hom(A,B)^\cdot : \quad \hom(A,B)^p := \{\text{graded homo's } A \to B \text{ of degree } -p\} =$$

$$= \prod_{i\in\mathbb{Z}} \hom(A_i, B_{i-p}).$$ ← correction from last lecture!

We define $\tilde{\delta}: \hom(A,B)^p \longrightarrow \hom(A,B)^{p+1}$ by the following formula:

$\partial_B\langle f, a\rangle = \langle \tilde{\delta} f, a\rangle + (-1)^p \langle f, \partial_A a\rangle$. This defines $\tilde{\delta}$.

Exc. 1) Show $\tilde{\delta}\circ\tilde{\delta} = 0$.

2) Let $f: A_\cdot \to B_\cdot$ be a graded homo. of deg. 0. Show that f is a cocycle $(\tilde{\delta} f = 0)$ iff f is a chain map. Show that f is a coboundary $(f = \tilde{\delta} h)$ iff f is chain homotopic to 0 (via the chain homotopy h).

3) Let $f, g: A_\cdot \longrightarrow B_\cdot$ be graded homo's of deg. 0 and assume f, g are chain maps (so, $f, g \in \hom^0(A,B)$ are cycles). Show that f & g are cohomologous in $\hom^\cdot(A,B)$ $\left(\text{i.e. } [f] = [g] \in H^0(\hom^\cdot(A,B), \tilde{\delta})\right)$ iff f is chain homotopic to g.

4) Generalize 2 & 3 for graded homo.'s of arbitrary degree.

Hint. Consider the shifted ch. complex $B[d]$, and endow it with the diff. $(-1)^d \partial_B$.

Cohomological cross product.

Fix a commutative ring R (with a unity).

Let X, Y be spaces. Let $\varphi \in S^p(X;R)$, $\psi \in S^q(Y;R)$ be cochains.

We'll define $\varphi \times \psi \in S^n(X\times Y;R)$, where $n = p+q$.

Write $\varphi : S_p(X) \longrightarrow R$, $\psi : S_q(Y) \longrightarrow R$.

Recall $\Theta : S_\cdot(X\times Y) \longrightarrow S_\cdot(X) \otimes_{\mathbb{Z}} S_\cdot(Y)$. Fix one such map.

Consider $\varphi \otimes \psi : S_p(X) \otimes_{\mathbb{Z}} S_q(Y) \longrightarrow R \otimes_{\mathbb{Z}} R \longrightarrow R$, where the last map is induced by the bilinear map $R\times R \longrightarrow R$, $(r_1, r_2) \longmapsto r_1 \cdot r_2$.

$\Rightarrow$ we get a cochain $\varphi \otimes \psi : \underbrace{\left(S_\cdot(X) \otimes_{\mathbb{Z}} S_\cdot(Y)\right)_{p+q}}_{\left(\bigoplus_{p'+q'=p+q} S_{p'}(X) \otimes_{\mathbb{Z}} S_{q'}(Y)\right)} \longrightarrow R$

by defining $\varphi \otimes \psi$ to be 0

$\forall\ p', q'$ s.t. $p'+q' = p+q$

but $(p', q') \neq (p, q)$.

Fix $\Theta : S_\cdot(X\times Y) \longrightarrow S_\cdot(X) \otimes_{\mathbb{Z}} S_\cdot(Y)$

Define $\varphi \times \psi \in S^{p+q}(X\times Y;R)$ by $\boxed{\varphi \times \psi := (\varphi \otimes \psi) \circ \Theta}$

Note: $\times$ is natural w.r.t. maps $X \to X'$, $Y \to Y'$, b.c. Θ has this property.

A more explicit formula. Let $c \in S_n(X \times Y)$, where $n = p+q$.

$$\Theta(c) = \sum_{r+s=n} \sum_{i,j} a_i^r \otimes b_j^s, \text{ with } a_i^r \in S_r(X),\ b_j^s \in S_s(Y).$$

$$(\varphi \times \psi)(c) = (-1)^{p \cdot q} \sum_{i,j} \varphi(a_i^p) \cdot \psi(b_j^q). \qquad \text{(Koszul sign conventions!)}$$

Claim. $\delta(\varphi \times \psi) = \delta\varphi \times \psi + (-1)^{|\varphi|} \varphi \times \delta\psi$. In other words, the map $S^p(X;R) \otimes_{\mathbb{Z}} S^q(Y;R) \longrightarrow S^{p+q}(X \times Y; R)$ induced by $(\varphi, \psi) \longmapsto \varphi \times \psi$ is a chain map. (w.r.t. the diff. $\delta_X \otimes id + id \otimes \delta_Y$ and $\delta_{X \times Y}$).

Remark. The map $(\varphi, \psi) \longmapsto \varphi \times \psi$ is bilinear over R, so it induces a map of R-modules $S^p(X;R) \otimes_R S^q(Y;R) \longrightarrow S^{p+q}(X \times Y; R)$.

Cor. The cochain level $\times$ product induces a product

$$H^p(X;R) \underset{R}{\otimes} H^q(Y;R) \xrightarrow{\times} H^{p+q}(X \times Y; R)$$ which is independent of the particular choice of Θ.

Proof. We have an algebraic map

$$\underbrace{H^p(S^\cdot(X;R)) \underset{R}{\otimes} H^q(S^\cdot(Y;R))}_{\substack{\shortparallel \\ H^p(X;R) \underset{R}{\otimes} H^q(Y;R)}} \xrightarrow{h} H^{p+q}\left(S^\cdot(X;R) \underset{R}{\otimes} S^\cdot(Y;R)\right) \xrightarrow{\times} \underbrace{H^{p+q}(S^\cdot(X\times Y;R))}_{\substack{\shortparallel \\ H^{p+q}(X\times Y;R)}}$$

The composition gives the desired map.

The independence of the specific choice of Θ follows from the fact that Θ is unique up to ch. homotopy. $\blacksquare$

The Kronecker product/pairing.

$X=$ space, $G=$ ab. group. $\Rightarrow$ $H^p(X;G)\otimes H_p(X)\longrightarrow G$.

Let $\alpha\in H^p(X;G)$, $a\in H_p(X)$. choose a cocycle $\varphi: S_p(X)\longrightarrow G$ with $\alpha=[\varphi]$, and a cycle $c\in S_p(X)$ with $a=[c]$.

$\langle\alpha,a\rangle := \varphi(c)$. <u>Exc.</u> show that the Kronecker pairing is well defined.

<u>Prop.</u> Let R be a ring. Let $\varphi\in S^\cdot(X;R)$, $\psi\in S^\cdot(Y;R)$ be cocycles of pure degree, and $a\in S_\cdot(X)$, $b\in S_\cdot(Y)$ be two cycles of pure deg.

Then $\langle\varphi\times\psi,\ a\times b\rangle = (-1)^{|\psi|\cdot|a|}\langle\varphi,a\rangle\cdot\langle\psi,b\rangle$. Here we use the convention that for a cochain $f: S_r(Z)\longrightarrow R$ and a chain $c\in S_s(Z)$ we have $\langle f,c\rangle = 0$ whenever $r\neq s$.

<u>Proof.</u> $\langle\varphi\times\psi,\ a\times b\rangle = \langle(\varphi\otimes\psi)\circ\Theta,\ a\times b\rangle = (\varphi\otimes\psi)\circ\Theta\circ\underset{\substack{\uparrow\\ \text{cross prod}}}{\times}(a\otimes b)$ $\quad(*)$

We know $\Theta\circ\times = id + D\partial_\otimes + \partial_\otimes D$ for some map $(**)$

$$D : S_\cdot(X)\otimes S_\cdot(Y) \longrightarrow \big(S_\cdot(X)\otimes S_\cdot(Y)\big)[1] \qquad (D \text{ is a ch. homotopy}).$$

Substitute (**) into (*) and get:

$$\langle \varphi\times\psi, a\times b\rangle = (\varphi\otimes\psi)\Big(a\otimes b + D\underbrace{\partial_\otimes(a\otimes b)}_{\substack{=0\\ \text{b.c. } a\,\&\,b\\ \text{are cycles}}} + \partial_\otimes D(a\otimes b)\Big) =$$

$$= (-1)^{|\psi|\cdot|a|}\langle\varphi,a\rangle\cdot\langle\psi,b\rangle \pm \underbrace{\delta_\otimes(\varphi\otimes\psi)}_{\substack{=0\\ \text{b.c. } \varphi\,\&\,\psi\\ \text{are cocycles.}}}\big(D(a\otimes b)\big) = (-1)^{|\psi|\cdot|a|}\langle\varphi,a\rangle\cdot\langle\psi,b\rangle.$$

$\blacksquare$

The unit (or unity) in cohomology. X = space, R = ring.

Denote by $1 \in H^0(X;R)$ the cohomology class of the cocycle

$\varepsilon : S_0(X) \longrightarrow R$ (the augmentation) that sends every

0-simplex $x \in X$ to $1 \in R$.

Exc. The class 1 is preserved by maps, i.e. if $f : X \longrightarrow Y$

then $f^*(1_Y) = 1_X$. (Notation: sometimes we write $1_X \in H^0(X;R)$ instead of 1)

Denote by $\mathbb{P}$ = the one-point space.

Prop. The composition $S.(X \times \mathbb{P}) \xrightarrow{\Theta} S.(X) \otimes S.(\mathbb{P}) \xrightarrow{id \otimes \varepsilon} S.(X) \otimes \mathbb{Z} \xrightarrow{\cong} S.(X)$

aug. for coeffs. in $\mathbb{Z}$

is naturally ch. homotopic to the ch. map $S.(X \times \mathbb{P}) \longrightarrow S.(X)$ induced by the identif. $\tau : X \times \mathbb{P} \longrightarrow X$, $\tau(x, p) = x$.

w.r.t. maps $X \longrightarrow X'$.

Proof: Can be done using acyclic models.

Consider the composition of maps from the prop. $S_\cdot(X\times P) \longrightarrow S_\cdot(X)$.

Dualize it : $S^\cdot(X;R) \longrightarrow S^\cdot(X\times P;R)$. short calcul. $\Rightarrow$ $\varphi \longmapsto \varphi\times\varepsilon$.

By the proposition we get: $\alpha\times 1_P = \tau^*\alpha \quad \forall\ \alpha\in H^p(X;R)$.

<u>Prop.</u> Let X, Y be spaces. Denote by
$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$
the projections.

Then $\forall\ \alpha\in H^p(X;R)$, $\beta\in H^q(Y;R)$ we have

$$\alpha\times 1_Y = \pi_X^*\alpha\,, \quad 1_X\times\beta = \pi_Y^*\beta.$$

<u>Proof.</u> We'll omit R from the notation. Consider $X\times Y \xrightarrow{id\times c_P} X\times P$, where $c_P : Y\to P$ is the const. map.

We have the following comm. diag.

$$\begin{tikzcd}
\alpha\otimes 1 \in H^p(X)\otimes\mathbb{Z} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \alpha\otimes 1_P \in H^p(X)\otimes H^0(P) \arrow[r, "id\otimes c_P^*"] \arrow[d, "\times"] & \alpha\otimes 1_Y \in H^p(X)\otimes H^0(Y) \arrow[d, "\times"] \\
\alpha \in H^p(X) \arrow[r, "\tau^*"] \arrow[rr, bend right, "(\tau\circ(id\times c_P))^*"'] & \alpha\times 1_P \in H^p(X\times P) \arrow[r, "(id\times c_P)^*"] & H^p(X\times Y) \ni \alpha\times 1_Y
\end{tikzcd}$$

commut. of the right-hand square is b.c. $\times$ is natural w.r.t. maps between spaces.

The left-hand square commut. b.c. $\alpha\times 1_P = \tau^*\alpha$

It follows that $(\tau\circ(id\times c_P))^*\alpha = \alpha\times 1_Y$.

But $\tau\circ(id\times c_P): X\times Y \longrightarrow X$ is exactly π_X. $\Longrightarrow \pi_X^*\alpha = \alpha\times 1_Y$.

The proof of the 2'nd identity is similar. $\blacksquare$

Cross product for relative cohomology.

X = space, $A \subset X$ subspace. R = ring. Recall $S^p(X,A;R) = \hom(S_p(X,A),R) =$

$= \hom\left(S_p(X)/S_p(A), R\right)$

So we can identify $S^p(X,A;R) = \ker\left(S^p(X;R) \longrightarrow S^p(A;R)\right)$.

$$\phi \longmapsto \phi|_{S_p(A)}$$

Let Y be another space.

Fix a Θ-map. We have the following commut. diag.

$$\begin{tikzcd} S_\cdot(A\times Y) \arrow[r, "\Theta"] \arrow[d, "\text{incl.}"] & S_\cdot(A)\otimes S_\cdot(Y) \arrow[d, "\text{incl.}"] \\ S_\cdot(X\times Y) \arrow[r, "\Theta"] & S_\cdot(X)\otimes S_\cdot(Y) \end{tikzcd}$$

(Note that both vert. maps are injective)

<u>Claim</u> If $\varphi \in S^p(X,A;R)$, $\psi \in S^q(Y;R)$ then

$$\varphi \times \psi \in S^{p+q}(X\times Y, A\times Y; R).$$

<u>Proof.</u> Let $c \in S_{p+q}(A\times Y)$. We have

$$(\varphi\times\psi)(c) = (\varphi\otimes\psi)\circ\Theta(c) = 0 \quad \text{b.c. } \varphi|_{S_p(A)} \equiv 0.$$

∎

Conclusion. The following diag. commutes.

$$\begin{tikzcd} H^p(X,A;R)\otimes H^q(Y;R) \arrow[r, "\times"] \arrow[d, "j^*\otimes id"'] & H^{p+q}((X,A)\times Y;R) \arrow[d, "k^*"] \\ H^p(X;R)\otimes H^q(Y;R) \arrow[r, "\times"] & H^{p+q}(X\times Y;R) \end{tikzcd}$$

$\Big((X,A)\times Y := (X\times Y, A\times Y)\Big)$

where $j: X \longrightarrow (X,A)$, $k: X\times Y \longrightarrow (X\times Y, A\times Y)$ are the obv. inclusions.

Prop. The following diag. commutes.

$$\begin{tikzcd} H^p(A)\otimes H^q(Y) \arrow[r, "\times"] \arrow[d, "\delta^*\otimes id"'] & H^{p+q}(A\times Y) \arrow[d, "\delta^*"] \\ H^{p+1}(X,A)\otimes H^q(Y) \arrow[r, "\times"] & H^{p+q+1}((X,A)\times Y) \end{tikzcd}$$

(works with coeffs. in R.)

If $B \subset Y$ is a subspace, then the relative $\times$ product gives

$$H^p(X) \otimes H^q(Y,B) \xrightarrow{\times} H^{p+q}(X \times (Y,B)),$$ however the diag.

$$\begin{tikzcd}
H^p(X) \otimes H^q(B) \arrow[rr, "\times"] \arrow[d, "id \otimes \delta^*"'] \arrow[drr, phantom, "© / (-1)^p"] & & H^{p+q}(X \times B) \arrow[d, "\delta^*"] \\
H^p(X) \otimes H^{q+1}(Y,B) \arrow[rr, "\times"] & & H^{p+q+1}(X \times (Y,B))
\end{tikzcd}$$

commutes only up to $(-1)^p$ sign.

Lecture #8B.

Prop. The following diag. commutes.

$$\begin{tikzcd} H^p(A)\otimes H^q(Y) \arrow[r, "\times"] \arrow[d, "\delta^*\otimes id"'] & H^{p+q}(A\times Y) \arrow[d, "\delta^*"] \\ H^{p+1}(X,A)\otimes H^q(Y) \arrow[r, "\times"] & H^{p+q+1}((X,A)\times Y) \end{tikzcd}$$

(works with coeffs. in R.)

Proof. Let $\varphi\in S^p(A)$, $\psi\in S^q(Y)$ be cocycles (i.e. $\delta\varphi=0$, $\delta\psi=0$).

Extend φ to a cochain $\tilde\varphi\in S^p(X)$.

clearly $\delta\tilde\varphi\Big|_{S_{p+1}(A)}\equiv 0$ b.c. $\delta\varphi=0$,

$$\begin{tikzcd} & & \tilde\varphi \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \varphi \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\ 0 \arrow[r] & S^p(X,A) \arrow[r] \arrow[d, "\delta"] & S^p(X) \arrow[r] \arrow[d, "\delta"] & S^p(A) \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & S^{p+1}(X,A) \arrow[r] & S^{p+1}(X) \arrow[r] & S^{p+1}(A) \arrow[r] & 0 \\ & \exists\ \tilde\varphi' \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto, dashed] & \delta\tilde\varphi \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & 0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \end{tikzcd}$$

hence we can view $\delta\tilde\varphi$ as an element of $S^{p+1}(X,A)$ (denoted in the diag. by $\tilde\varphi'$).

$$\delta^*[\varphi]=[\delta\tilde\varphi]\in H^{p+1}(X,A).$$

$$\begin{tikzcd} H^p(A)\otimes H^q(Y) \ni [\varphi]\otimes[\psi] \arrow[d, "\delta^*\otimes id"'] & \\ H^{p+1}(X,A)\otimes H^q(Y) \arrow[r, "\times"] & H^{p+q+1}((X,A)\times Y) \\ {[\delta\tilde\varphi]\otimes[\psi]} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & {[\delta\tilde\varphi\times\psi]} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

Consider the other composition in the diag.

The cochain $\tilde{\varphi} \times \psi \in S^{p+q}(X \times Y)$ extends $\varphi \times \psi \in S^{p+q}(A \times Y)$

(this follows from naturality of $\times$ w.r.t. maps; In this case $A \xrightarrow{i} X$, $Y \xrightarrow{id} Y$.)

$$\Rightarrow \delta^*([\varphi \times \psi]) = [\delta(\tilde{\varphi} \times \psi)] \in H^{p+q+1}(X \times Y, A \times Y).$$

But $\delta(\tilde{\varphi} \times \psi) = \delta\tilde{\varphi} \times \psi$ b.c. $\delta\psi = 0$. $\Rightarrow \delta^*[\varphi \times \psi] = [\delta\tilde{\varphi} \times \psi]$.

$\blacksquare$

<u>Important remark.</u> Let X = path-connected, $\phi \neq A \subset X$ subspace.

There is <u>no</u> element 1 in $H^0(X, A)$!

<u>Commutativity of the cross product.</u> (Fix a ring R for coeffs. omit from not.)

Let $\alpha \in H^p(X)$, $\beta \in H^q(Y)$. <u>Q.</u> what's the relation betwee $\alpha \times \beta \in H^{p+q}(X \times Y)$ and $\beta \times \alpha \in H^{p+q}(Y \times X)$?

We'll identify $X \times Y \approx Y \times X$ using the obvious map $T: X \times Y \longrightarrow Y \times X$

$(x,y) \longmapsto (y,x)$.

Consider the following diag.:

$$\begin{tikzcd}
S.(X \times Y) \arrow[r, "\Theta_{X,Y}"] \arrow[d, "T_c"'] & S.(X) \otimes S.(Y) \\
S.(Y \times X) \arrow[r, "\Theta_{Y,X}"] & S.(Y) \otimes S.(X) \arrow[u, "\tau"']
\end{tikzcd}$$

$\tau(b \otimes a) := (-1)^{|b||a|} a \otimes b.$

<u>Exc.</u> τ is a chain map.

Consider the composition $\tau \circ \Theta_{Y,X} \circ T_c$. This is a ch. map $S.(X \times Y) \to S.(X) \otimes S.(Y)$. It is natural w.r.t. maps $X \to X'$, $Y \to Y'$, and in degree 0 this map does $(x,y) \longmapsto x \otimes y$.

By a previous Thm., $\exists$ a chain homotopy $\tau\circ\Theta_{Y,X}\circ T_c \simeq \Theta_{X,Y}$, i.e.
$\exists$ an operator $D: S_\cdot(X\times Y)\longrightarrow \big(S_\cdot(X)\otimes S_\cdot(Y)\big)[1]$ s.t.

$$\tau\circ\Theta_{Y,X}\circ T_c - \Theta_{X,Y} = D\circ\partial + \partial_\otimes\circ D.$$

We now pass to cohomology. Let $f\in S^p(X)$, $g\in S^q(Y)$ be cocycles.
Let's calculat $T^*\big([g]\times[f]\big)$:

$$T^*\big([g]\times[f]\big) = T^*\big([g\times f]\big) = T^*\big[(g\otimes f)\circ\Theta_{Y,X}\big] = \big[(g\otimes f)\circ\Theta_{Y,X}\circ T_c\big] =$$

$$= (-1)^{p\cdot q}\big[(f\otimes g)\circ\tau\circ\Theta_{Y,X}\circ T_c\big] = (-1)^{p\cdot q}\big[(f\otimes g)\circ\Theta_{X,Y} + (f\otimes g)\circ(D\partial+\partial_\otimes D)\big]$$

The 2'nd term in the $[\dots]$: it is a coboundary b.c. f,g are cocycles
so $(f\otimes g)\circ\partial_\otimes = 0$, and $(f\otimes g)\circ D\partial = \pm\delta\big((f\otimes g)\circ D\big) =$ coboundary.

$$\Rightarrow\quad T^*\big([g]\times[f]\big) = (-1)^{pq}[f]\times[g].$$

Prop. $\forall\ \alpha\in H^p(X;R)$, $\beta\in H^q(Y;R)$
we have $\alpha\times\beta = (-1)^{p\cdot q}\,T^*(\beta\times\alpha)$.

Associativity of the cross product.

Prop. Let X, Y, Z be spaces, $\alpha \in H^*(X)$, $\beta \in H^*(Y)$, $\gamma \in H^*(Z)$.

Then $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma) \in H^*(X \times Y \times Z)$. Similarly, if $a \in H_*(X)$, $b \in H_*(Y)$, $c \in H_*(Z)$, then $(a \times b) \times c = a \times (b \times c) \in H_*(X \times Y \times Z)$.

Proof. We have two ch. maps

$$S_\cdot(X) \otimes S_\cdot(Y) \otimes S_\cdot(Z) \longrightarrow S_\cdot(X \times Y \times Z),$$

endowed with the obvious diff.

the 1'st is: $S_\cdot(X) \otimes S_\cdot(Y) \otimes S_\cdot(Z) \xrightarrow{\times \otimes id} S_\cdot(X \times Y) \otimes S_\cdot(Z) \xrightarrow{\times} S_\cdot(X \times Y \times Z)$

the 2'nd is: $S_\cdot(X) \otimes S_\cdot(Y) \otimes S_\cdot(Z) \xrightarrow{id \otimes \times} S_\cdot(X) \otimes S_\cdot(Y \times Z) \xrightarrow{\times} S_\cdot(X \times Y \times Z)$.

Both ch. maps are natural in X, Y, Z and both maps equal to

$$x \otimes y \otimes z \longmapsto (x, y, z) \quad \text{in deg. } 0.$$

Using an argument based on ~~ay~~ acyclic models one shows these two ch. maps are ch. homotopic. $\Rightarrow$ both maps induce the same map in homology.

The statement in cohomology: consider two ch. maps

$$S_\cdot(X\times Y\times Z) \longrightarrow S_\cdot(X)\otimes S_\cdot(Y)\otimes S_\cdot(Z)$$

$$\begin{tikzcd}
 & S_\cdot(X\times Y)\otimes S_\cdot(Z) \arrow[dr, "\Theta_{X,Y}\otimes id"] & \\
S_\cdot(X\times Y\times Z) \arrow[ur, "\Theta_{X\times Y,Z}"] \arrow[dr, "\Theta_{X,Y\times Z}"'] \arrow[rr, bend left] \arrow[rr, bend right] & & S_\cdot(X)\otimes S_\cdot(Y)\otimes S_\cdot(Z) \\
 & S_\cdot(X)\otimes S_\cdot(Y\times Z) \arrow[ur, "id\otimes\Theta_{Y,Z}"'] &
\end{tikzcd}$$

Let f, g, h be cocycles representing α, β, γ respectively.

$$(f\times g)\times h = ((f\times g)\otimes h)\circ\Theta_{X\times Y,Z} = (((f\otimes g)\circ\Theta_{X,Y})\otimes h)\circ\Theta_{X\times Y,Z} =$$

$$= (f\otimes g\otimes h)\circ((\Theta_{X,Y}\otimes id)\circ\Theta_{X\times Y,Z}) = (f\otimes g\otimes h)\circ(\text{upper-composition}).$$

$f\times(g\times h) = \ldots = (f\otimes g\otimes h)\circ(\text{lower-composit.})$. But and are ch. homotopic. ∎

The cup product.

X = space. Fix a ring of coefficients R.

We'll define an operation $H^p(X;R) \otimes_R H^q(X;R) \xrightarrow{\cup} H^{p+q}(X;R)$,

$$\alpha \otimes \beta \longmapsto \alpha \cup \beta$$

Consider the diagonal map $d : X \longrightarrow X \times X$, $d(x) = (x,x)$.

Define $\alpha \cup \beta := d^*(\alpha \times \beta)$. The operation $\cup$ is independent of the choice of Ⓗ, in chomology.

Properties of $\cup$.

1) $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$, $\forall$ $\alpha \in H^p(X;R)$, $\beta \in H^q(X;R)$
2) $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$ $\forall$ $\alpha, \beta, \gamma \in H^*(X;R)$
3) $\alpha \cup 1 = 1 \cup \alpha = \alpha$.
4) Let X, Y be spaces, $\alpha_1, \alpha_2 \in H^*(X;R)$, $\beta_1, \beta_2 \in H^*(Y;R)$ be elements of pure deg.

Then $(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) = (-1)^{|\beta_1|\cdot|\alpha_2|} (\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)$.

5) Let $\pi_X : X \times Y \longrightarrow X$, $\pi_Y : X \times Y \longrightarrow Y$ be the projections.

$\Rightarrow \forall\ \alpha \in H^*(X;R), \beta \in H^*(Y)$ we have $\pi_X^* \alpha \cup \pi_Y^* \beta = \alpha \times \beta$.

(So, the cup product determines the cross product and vice-versa)

Put $H^*(X;R) = \bigoplus_{i \geq 0} H^i(X;R)$. Extend the cup product linearly to $H^*(X;R) \otimes_R H^*(X;R) \xrightarrow{\cup} H^*(X;R)$.

<u>Cor.</u> The cup prod. makes $H^*(X;R)$ ~~into~~ a graded ring with a unity which $1 \in H^0(X;R)$. It is called the cohomology ring of X. This ring is not commutative, but ~~sup~~ graded-commutative.

Lecture #9A.

The cup product.

X = space. Fix a ring of coefficients R.

We'll define an operation $H^p(X;R) \otimes_R H^q(X;R) \xrightarrow{\cup} H^{p+q}(X;R)$,

$$\alpha \otimes \beta \longmapsto \alpha \cup \beta$$

Consider the diagonal map $d: X \longrightarrow X \times X$, $d(x) = (x, x)$.

Define $\alpha \cup \beta := d^*(\alpha \times \beta)$. The operation $\cup$ is independent of the choice of Ⓗ, in chomology.

Properties of $\cup$.

1) $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$, $\forall$ $\alpha \in H^p(X;R)$, $\beta \in H^q(X;R)$

2) $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$ $\forall$ $\alpha, \beta, \gamma \in H^*(X;R)$

3) $\alpha \cup 1 = 1 \cup \alpha = \alpha$.

4) Let X, Y be spaces, $\alpha_1, \alpha_2 \in H^*(X;R)$, $\beta_1, \beta_2 \in H^*(Y;R)$ be elements of pure deg.

Then $(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) = (-1)^{|\beta_1| \cdot |\alpha_2|} (\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)$.

5) Let $\pi_X : X \times Y \longrightarrow X$, $\pi_Y : X \times Y \longrightarrow Y$ be the projections.

$\Rightarrow \forall\ \alpha \in H^*(X;R)$, $\beta \in H^*(Y)$ we have $\pi_X^* \alpha \cup \pi_Y^* \beta = \alpha \times \beta$.

(So, the cup product determines the cross product and vice-versa)

Put $H^*(X;R) = \bigoplus_{i \geq 0} H^i(X;R)$. Extend the cup product linearly

to $H^*(X;R) \otimes_R H^*(X;R) \xrightarrow{\cup} H^*(X;R)$.

<u>Cor.</u> The cup prod. makes $H^*(X;R)$ a graded ring with a unity which $1 \in H^0(X;R)$. It is called the cohomology ring of X.
This ring is not commutative, but graded-commutative.

We'll denote also by $\cup$ "the" chain-level oper. $S^p(X;R) \underset{R}{\otimes} S^q(X;R) \xrightarrow{\cup} S^{p+q}(X;R)$

$(*)$ $a \otimes b \longmapsto a \cup b := d^c(a \times b)$.

$\langle a \cup b, c \rangle = \langle a \otimes b, \oplus d_c(c) \rangle$

$\forall \; c \in S_{p+q}(X)$

<u>Prop.</u> $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^{|\varphi|} \varphi \cup \delta\psi$

<u>Proof.</u> Exe.

The prop says that the ch. level $\cup$ product $(*)$ is a chain map.

<u>Proof of the properties of $\cup$ product.</u> (we omit R from notat.)

1) $\alpha \cup \beta = (-1)^{p \cdot q} \beta \cup \alpha \quad \forall \; \alpha \in H^p(X), \beta \in H^q(X)$.

$T : X \times X \longrightarrow X \times X$

$(x, y) \longmapsto (y, x)$

$\beta \cup \alpha = d^*(\beta \times \alpha) = (-1)^{pq} d^* T^*(\alpha \times \beta)$

But $d^* \circ T^* = (T \circ d)^* = d^*$ b.c. $T \circ d = d$.

So $\beta \cup \alpha = \ldots = (-1)^{pq} d^*(\alpha \times \beta) = (-1)^{pq} \alpha \cup \beta$.

2) $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma) \quad \forall \quad \alpha, \beta, \gamma \in H^*(X)$.

$$\begin{aligned} d_X : X &\longrightarrow X \times X \\ x &\longmapsto (x, x) \end{aligned}$$

$$(\alpha \cup \beta) \cup \gamma = d_X^*\big((\alpha \cup \beta) \times \gamma\big) = d_X^*\big(d_X^*(\alpha \times \beta) \times \gamma\big) =$$

$$= d_X^* \circ (d_X \times id_X)^*\big((\alpha \times \beta) \times \gamma\big) = \big((d_X \times id_X) \circ d_X\big)^* (\alpha \times \beta \times \gamma)$$

But $(d_X \times id_X) \circ d_X = \big(X \ni x \longmapsto (x, x, x) \in X \times X \times X \big)$

Similarly: $\alpha \cup (\beta \cup \gamma) = d_X^*\big(\alpha \times (\beta \cup \gamma)\big) = d_X^*\big(\alpha \times d_X^*(\beta \times \gamma)\big) =$

$$= d_X^* (id_X \times d_X)^* (\alpha \times \beta \times \gamma) = \big((id_X \times d_X) \circ d_X\big)^* (\alpha \times \beta \times \gamma)$$

and again $(id_X \times d_X) \circ d_X = \big(X \ni x \longmapsto (x, x, x) \in X \times X \times X \big)$.

3) $\alpha \cup 1 = d^*(\alpha \times 1) = d^* pr_1^* \alpha = (pr_1 \circ d)^* \alpha = (id)^* \alpha = \alpha$.

The proof that $1 \cup \alpha = \alpha$ is similar.

$$\begin{array}{c} X \times X \\ \big\downarrow pr_1 \\ X \end{array} \quad \text{proj. on 1'st factor}$$

4) X, Y spaces. $\alpha_1, \alpha_2 \in H^*(X)$, $\beta_1, \beta_2 \in H^*(Y)$ of pure degree.

$$(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) = (-1)^{|\beta_1|\cdot|\alpha_2|} (\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2).$$

<u>Proof.</u> Put $d_X : X \longrightarrow X \times X$, $d_Y : Y \longrightarrow Y \times Y$, $d_{X\times Y} : X \times Y \longrightarrow X \times Y \times X \times Y$ to be the diag. maps. Denote by

$$T' : X \times X \times Y \times Y \longrightarrow X \times Y \times X \times Y$$
$$(x_1, x_2, y_1, y_2) \longmapsto (x_1, y_1, x_2, y_2).$$

Note that $d_{X\times Y} = T' \circ (d_X \times d_Y)$.

$$(\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) = d^*_{X\times Y}(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) = (d_X \times d_Y)^* T'^* (\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) =$$
$$= (-1)^{|\beta_1|\cdot|\alpha_2|} (d_X \times d_Y)^* (\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2) = (-1)^{|\beta_1|\cdot|\alpha_2|} d_X^*(\alpha_1 \times \alpha_2) \times d_Y^*(\beta_1 \times \beta_2) =$$
$$= (-1)^{|\beta_1|\cdot|\alpha_2|} (\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2).$$

5) $$\pi_X^*(\alpha) \cup \pi_Y^*(\beta) = (\alpha \times 1_Y) \cup (1_X \times \beta) \underset{\substack{\uparrow \\ \text{by property 4}}}{=} (\alpha \cup 1_X) \times (1_Y \cup \beta) = \alpha \times \beta.$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \text{proj.} \\ X & & Y \end{tikzcd}$$

Naturality (on the chain level). (we omit R from notat.)

Prop. Let $f: X \to Y$, $\varphi \in S^p(Y)$, $\psi \in S^q(Y)$.

Then $f^c(\varphi \cup \psi) = f^c(\varphi) \cup f^c(\psi)$.

Cor. $f^*: H^*(Y) \longrightarrow H^*(X)$ is a map of rings, i.e. it respects the products.

Proof of Prop. $f^c(\varphi \cup \psi) = f^c d_Y^c (\varphi \times \psi) = (d_Y \circ f)^c (\varphi \times \psi)$ $\quad (*)$

But $d_Y \circ f = \left(X \xrightarrow{d_X} X \times X \xrightarrow{f \times f} Y \times Y \right)$. So from $(*)$ we get

$$f^c(\varphi \cup \psi) = \left((f\times f)\circ d_X\right)^c (\varphi \times \psi) = d_X^c (f \times f)^c (\varphi \times \psi) \underset{\substack{\uparrow \\ \text{naturality} \\ \text{of } \times}}{=}$$

$$= d_X^c \left(f^c(\varphi) \times f^c(\psi)\right) = f^c(\varphi) \cup f^c(\psi).$$

$\blacksquare$

The relative case.

X = space, $A \subset X$ subspace. $i: A \to X$ incl. Fix coeff. ring R, and omit from notat.

Let $\varphi \in S^p(X)$, $\psi \in S^q(X)$. By naturality, if $\varphi\big|_{S_p(A)} \equiv 0$, then

$$i^c(\varphi \cup \psi) = \underbrace{i^c(\varphi)}_{0} \cup i^c(\psi) = 0. \Rightarrow (\varphi \cup \psi)\big|_{S_{p+q}(A)} \equiv 0.$$

Similarly, if $B \subset X$ is a subspace and $\psi\big|_{S_q(B)} \equiv 0 \Rightarrow (\varphi \cup \psi)\big|_{S_{p+q}(B)} \equiv 0$.

Conclusion. The $\cup$ product induces

$$H^p(X,A) \otimes H^q(X) \xrightarrow{\cup} H^{p+q}(X,A)$$

$$H^p(X) \otimes H^q(X,B) \xrightarrow{\cup} H^{p+q}(X,B).$$

If $A, B \subset X$ are <u>open</u>, then we also have

$$H^p(X,A) \otimes H^q(X,B) \xrightarrow{\cup} H^{p+q}(X, A \cup B).$$

Proof of the last statement. Let $\varphi \in S^p(X,A)$, $\psi \in S^q(X,B)$.

View φ & ψ as maps $S_p(X) \xrightarrow{\varphi} R$ with $\varphi|_{S_p(A)} \equiv 0$

and $S_q(X) \xrightarrow{\psi} R$ with $\psi|_{S_q(B)} \equiv 0$.

We've seen that $\varphi \cup \psi$ is 0 on $S_{p+q}(A)$ as well as on $S_{p+q}(B)$.

Let $S_\cdot^{A,B} \subset S_\cdot(A \cup B)$ be the subcomplex generated by the chains on A and the chains on B. (We've seen this is a subcomplex). We also know that the incl. $S_\cdot^{A,B} \subset S_\cdot(A \cup B)$ induces an iso. in homology.

The ch. level $\cup$-prod. gives:

$$(*) \qquad S^p(X,A) \otimes S^q(X,B) \xrightarrow{\ \cup\ } \left(S_{p+q}(X)/S_\cdot^{A,B}\right)^* \leftarrow \hom(-,R).$$

Consider the SES of ch. complexes:

$$\begin{array}{ccccccccc}
0 & \to & S_\cdot^{A,B} & \to & S_\cdot(X) & \to & S_\cdot(X)/S_\cdot^{A,B} & \to & 0 \\
 & & \downarrow i & & \downarrow id & & \downarrow & & \\
0 & \to & S_\cdot(A \cup B) & \to & S_\cdot(X) & \to & S_\cdot(X)/S_\cdot(A \cup B) & \to & 0
\end{array}$$

The vert. arrows come from inclusions

We'll dualize ($\hom(-,R)$) these sequences and obtain

$$\begin{tikzcd}
0 & (S_\cdot^{A,B})^* \arrow[l] & S^\cdot(X) \arrow[l] & (S_\cdot(X)/S_\cdot^{A,B})^* \arrow[l] & 0 \arrow[l] \\
0 & S^\cdot(A\cup B) \arrow[l] \arrow[u, "i^*"] & S^\cdot(X) \arrow[l] \arrow[u, "id"] & S^\cdot(X, A\cup B) \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

The exactness is preserved b.c. all the groups dualized are free.

in deg. i:

$\left\{ \begin{array}{l} \varphi : S_i(X) \longrightarrow R : \\ \varphi(\sigma) = 0 \ \forall \text{ i-simplices } \sigma \\ \text{that are either in } A \\ \text{or in } B \end{array} \right\}$

The map i^* is also a quasi-iso. (i.e. it induces an iso. in cohomology) - this follows from UCT.

Take the LES's induced by the above SES's + the 5-lemma and obtain $H^*\left(\left(S_\cdot(X)/S_\cdot^{A,B}\right)^*\right) \cong H^*(S^\cdot(X, A\cup B)) = H^*(X, A\cup B)$.

$\Rightarrow$ $\circledast$ gives us $H^p(X,A) \otimes H^q(X,B) \longrightarrow H^{p+q}(X, A\cup B)$.

An explicit chain-level formula for $\cup$-prod.

Consider the composit. of ch. maps

$$\Delta : \quad S_\bullet(X) \xrightarrow{d_c} S_\bullet(X \times X) \xrightarrow{\Theta} S_\bullet(X) \otimes S_\bullet(X). \qquad (1)$$

Note that in deg 0, Δ does $\Delta(x) = x \otimes x$ and also

$$f \cup g = (f \otimes g) \circ \Theta \circ d_c = (f \otimes g) \circ \underline{\Delta} \quad \forall \text{ coch. } f \,\&\, g.$$

<u>Def.</u> A diagonal approximation is a natural (in X) ch. map

$\Delta : S_\bullet(X) \longrightarrow S_\bullet(X) \otimes S_\bullet(X)$ that satisfies $\Delta(x) = x \otimes x$

$\forall$ 0-simplices $x \in X$.

<u>Example:</u> (1) above is a diag. approx.

<u>Thm.</u> Any two diag. approximations are ch. homotopic via a natural (in X) ch. homotopy.

<u>Proof.</u> Exc. Use acyclic models.

<u>Cor.</u> Let Δ be any diag. approx. Then $\forall$ cocycles $\varphi \in S^p(X)$, $\psi \in S^q(Y)$

we have $[\varphi] \cup [\psi] = \left[(\varphi \otimes \psi) \circ \Delta \right]$, where $(\varphi \otimes \psi) \circ \Delta$ is the cochain

$$\begin{tikzcd} S_{p+q}(X) \arrow[r, "\Delta"] \arrow[rrrr, bend right, "(\varphi \otimes \psi) \circ \Delta"'] & \left(S_\cdot(X) \otimes S_\cdot(X) \right)_{p+q} \arrow[r, "\text{proj.}"] & S_p(X) \otimes S_q(X) \arrow[r, "\varphi \otimes \psi"] & R \otimes R \arrow[r] & R \end{tikzcd}$$

Note that $(\varphi \otimes \psi) \circ \Delta$ is a cocycle b.c. φ & ψ are & Δ is a ch. map.

Lecture #9B.

An explicit chain-level formula for $\cup$-prod.

Def. A diagonal approximation is a natural (in X) ch. map $\Delta : S_\cdot(X) \longrightarrow S_\cdot(X) \otimes S_\cdot(X)$ that satisfies $\Delta(x) = x \otimes x$ $\forall$ 0-simplices $x \in X$.

Example. $\Delta : \left(S_\cdot(X) \xrightarrow{d_c} S_\cdot(X \times X) \xrightarrow{\Theta} S_\cdot(X) \otimes S_\cdot(X) \right)$ is a diag. approx.

Note that in deg 0, Δ does $\Delta(x) = x \otimes x$ and also

$$\boxed{f \cup g = (f \otimes g) \circ \Theta \circ d_c = (f \otimes g) \circ \Delta \quad \forall \text{ coch. } f \,\&\, g}$$

Thm. Any two diag. approximations are ch. homotopic via a natural (in X) ch. homotopy.

Cor. Let Δ be any diag. approx. Then $\forall$ cocycles $\varphi \in S^p(X)$, $\psi \in S^q(Y)$ we have $[\varphi] \cup [\psi] = [(\varphi \otimes \psi) \circ \Delta]$, where $(\varphi \otimes \psi) \circ \Delta$ is the cochain (actually cocycle!)

$$\begin{tikzcd} S_{p+q}(X) \arrow[r, "\Delta"] \arrow[rrrr, bend right, "(\varphi \otimes \psi) \circ \Delta"'] & (S_\cdot(X) \otimes S_\cdot(X))_{p+q} \arrow[r, "\text{proj.}"] & S_p(X) \otimes S_q(X) \arrow[r, "\varphi \otimes \psi"] & R \otimes R \arrow[r] & R \end{tikzcd}$$

The Alexander-Whitney diag. approx.

Let $0 \leq p \leq n$, $\sigma : \Delta^n \longrightarrow X$ a sing. n-simplex in X.

Define a new sing. p-simplex $\sigma\rfloor_p : \Delta^p \longrightarrow X$, called the front p-face of σ,

by $\sigma\rfloor_p([v_0 : \cdots : v_p]) := \sigma([v_0 : \_ : v_p : 0 : \cdots : 0])$, i.e. retrict σ to

the front p-face of Δ^n. Alternatively, let $F_p : \Delta^p \hookrightarrow \Delta^n$, $F_p(e_i) = e_i \ \forall\, 0 \leq i \leq p$,

then $\sigma\rfloor_p = \sigma \circ F_p$.

Similarly, for $0 \leq q \leq n$, define the back q-face of σ,

$${}_q\lfloor\sigma : \Delta^q \longrightarrow X, \quad \text{by} \quad {}_q\lfloor\sigma([u_0 : \_ : u_q]) = \sigma([0 : \cdots : 0 : u_0 : \_ : u_q]),$$

or alternatively ${}_q\lfloor\sigma = \sigma \circ B_q$, where $B_q : \Delta^q \hookrightarrow \Delta^n$ is the map

$$B_q(e_i) = e_{n-q+i} \quad \forall \quad 0 \leq i \leq q.$$

Define the A-W diag. approx: $\Delta^{AW} : S.(X) \longrightarrow S.(X) \otimes S.(X)$,

$$\forall\, \sigma : \Delta^n \to X, \quad \Delta^{AW}(\sigma) := \sum_{\substack{p+q=n \\ 0 \leq p,q}} \sigma\rfloor_p \otimes {}_q\lfloor\sigma \in \big(S.(X) \otimes S.(X)\big)_n.$$

More explicitely:

$$\overset{AW}{\Delta}(\sigma) = \sum_{p=0}^{n} \sigma\Big|_{[e_0:\,\_\,:e_p]} \otimes \sigma\Big|_{[e_p:\,\_\,:e_n]}$$

note the overlap at the vertex e_p

Exc. 1) $\overset{AW}{\Delta}$ is natural w.r.t. maps $X \longrightarrow Y$.

2) $\Delta^{AW}(x) = x \otimes x \quad \forall\, x \in X$, in degree 0.

Prop. Δ^{AW} is a chain map.

Some preparations for the proof.

Let $\sigma : \Delta^N \longrightarrow X$ be a sing. N-simplex and $0 \leq k \leq N$.

Recall the k-face of σ, $F^k\sigma : \Delta^{N-1} \longrightarrow X$, defined by

$$F^k\sigma := \sigma\Big|_{[e_0:\,\dots\,:\hat{e}_k:\,\dots\,:e_N]}.$$

For $N=0$ define $F^0\sigma = 0$ viewed as a chain in deg. -1.

$$\partial\sigma = \sum_{k=0}^{N} (-1)^k F^k\sigma.$$

Lemma. Let $\sigma : \Delta^n \longrightarrow X$ be a sing. n-simplex in X.

1) Let $0 \leq p, q$ with $p+q=n$. Then:

$$F^k\left(\sigma\rfloor_p\right) = \begin{cases} \left(F^k\sigma\right)\rfloor_{(p-1)} & 0 \leq k \leq p,\ 1 \leq p \\ 0 & k=0,\ p=0 \end{cases} \qquad\Bigg|\qquad F^k\left({}_q\lfloor\sigma\right) = \begin{cases} {}_{(q-1)}\lfloor F^{p+k}\sigma & 0 \leq k \leq q,\ 1 \leq q \\ 0 & k=0,\ q=0 \end{cases}$$

2) Let $0 \leq s, t$ s.t. $s+t=n-1$. Let $0 \leq k \leq n$. Then

$$\left(F^k\sigma\right)\rfloor_s = \begin{cases} \sigma\rfloor_s & 0 \leq s \leq k-1 \\ F^k\left(\sigma\rfloor_{s+1}\right) & k \leq s \end{cases} \qquad\Bigg|\qquad {}_t\lfloor F^k\sigma = \begin{cases} {}_t\lfloor\sigma & t \leq n-1-k \\ & (\Leftrightarrow k \leq s) \\ F^{k-s}\left({}_{t+1}\lfloor\sigma\right) & t \geq n-k \\ & (\Leftrightarrow s < k) \end{cases}$$

Proof. Exc.

<u>Proof of the prop.</u> We'll write Δ for Δ^{AW} in the proof.

Let $\sigma : \Delta^n \longrightarrow X$. $\quad \Delta(\partial\sigma) = \Delta\left(\sum_{i=0}^{n}(-1)^i F^i\sigma\right) = \sum_{i=0}^{n}(-1)^i \sum_{s+t=n-1} F^i\sigma\rfloor_s \otimes {}_t\lfloor F^i\sigma =$

$$= \sum_{i=0}^{n}\sum_{s=0}^{i-1}(-1)^i\, \sigma\rfloor_s \otimes F^{i-s}\left({}_{n-s}\lfloor\sigma\right) + \sum_{i=0}^{n}\sum_{s=i}^{n-1}(-1)^i F^i\left(\sigma\rfloor_{s+1}\right)\otimes {}_{(n-1-s)}\lfloor\sigma =$$

$$= \underbrace{\sum_{\substack{1\le j,\ 0\le s\\ s+j\le n}} (-1)^{s+j}\, \sigma\rfloor_s \otimes F^j\left({}_{n-s}\lfloor\sigma\right)}_{(*)_1} + \underbrace{\sum_{0\le i<p\le n}(-1)^i F^i\left(\sigma\rfloor_p\right)\otimes {}_{(n-p)}\lfloor\sigma}_{(*)_2}. \qquad (*)$$

put $j := i-s$ $\qquad\qquad$ $(p = s+1)$

Let's calculate $\partial_\otimes \Delta\sigma$:

$$\partial_\otimes\Delta\sigma = \partial_\otimes\left(\sum_{p+q=n}\sigma\rfloor_p \otimes {}_q\lfloor\sigma\right) = \sum_{p+q=n}\left(\partial(\sigma\rfloor_p)\otimes {}_q\lfloor\sigma + (-1)^p\,\sigma\rfloor_p\otimes\partial\left({}_q\lfloor\sigma\right)\right) =$$

$$= \underbrace{\sum_{p=0}^{n}\sum_{j=0}^{p}(-1)^j F^j\left(\sigma\rfloor_p\right)\otimes {}_q\lfloor\sigma}_{(**)_2} + \underbrace{\sum_{p=0}^{n}\sum_{l=0}^{n-p}(-1)^p(-1)^l\,\sigma\rfloor_p\otimes F^l\left({}_{n-p}\lfloor\sigma\right)}_{(**)_1} \qquad (**)$$

$(q = n-p)$

sum is in fact over all $0\le p,\ 0\le l$ s.t. $p+l\le n$.

$$(**)_2 - (*)_2 = \sum_{p=0}^{n} (-1)^p F^p\left(\sigma\rfloor_p\right) \otimes {}_{n-p}\lfloor\sigma = \sum_{p=0}^{n} (-1)^p \, \sigma\rfloor_{p-1} \otimes {}_{n-p}\lfloor\sigma \, . \qquad (\mathrm{I})$$

the only diff. is $j = p$ in $(**)_2$

for $p=0$ we have a 0-term.

$$(**)_1 - (*)_1 = \sum_{p=0}^{n} (-1)^p \, \sigma\rfloor_p \otimes {}_{(n-1-p)}\lfloor\sigma \, . \qquad (\mathrm{II})$$

the only diff. is for $l=0$ in $(**)_1$

for $p=n$, we have a 0-term

clearly $\underbrace{(\mathrm{I}) + (\mathrm{II})}_{\Delta\partial\sigma - \partial_{\otimes}\Delta\sigma} = 0$. ∎

<u>Cor.</u> The AW map Δ^{AW} is a diag. approx.

Cor. We can define (another) chain level cup prod. as follows:

Let $\varphi \in S^p(X;R)$, $\psi \in S^q(X;R)$, viewed as $\varphi : S_p(X) \longrightarrow R$, $\psi : S_q(X) \longrightarrow R$,

Put $n := p+q$. Define $\varphi \cup \psi : S_{p+q}(X) \longrightarrow R$

$$\langle \varphi \cup \psi, \sigma \rangle = \langle \varphi \otimes \psi, \Delta^{AW} \sigma \rangle = \langle \varphi \otimes \psi, \sum_{r+s=n} \sigma\rfloor_r \otimes {}_s\lfloor\sigma \rangle =$$

$$= (-1)^{pq} \varphi(\sigma\rfloor_p) \cdot \psi({}_q\lfloor\sigma), \quad \forall \; \sigma : \Delta^n \longrightarrow X.$$

Examples.

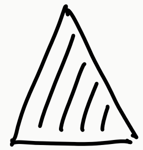
Let X be a (finite) CW-complex. We view the attaching cells of X as Δ^k rather than B^k. This is not a problem since $\exists$ a prefered class of homeo's $(B^k, \partial B^k) \xrightarrow{\approx} (\Delta^k, \partial \Delta^k)$ and they all restrict to homotopic homeo's $\partial B^k \xrightarrow{\approx} \partial \Delta^k$.

Consider the cellular ch. complex $C_\cdot^{cw}(X)$ of X.

$\exists$ an obvious inclusion map $i : C_\cdot^{cw}(X) \longrightarrow \mathbf{S}_\cdot(X)$. Problem: i is, in general, not a ch. map.

<u>Example.</u> $X = S^2$ with one 0-cell, one 2-cell (no 1-cells).

show that i is not a ch. map. (exc.)



We'll add now the following assumption:

1) $i : C_\cdot^{cw}(X) \longrightarrow S_\cdot(X)$ <u>is</u> a ch. map.

2) i induces an iso. in homology $i_* : H_*(C_\cdot^{cw}(X)) \xrightarrow{\cong} H_*(X)$.

<u>Claim.</u> Under the above assumptions, $i^* : S^\cdot(X;R) \longrightarrow C^\cdot_{cw}(X;R)$ induces an iso. in cohomology $i^* : H^*(X) \longrightarrow H^*(C^\cdot_{cw}(X;R))$. $\left(\begin{matrix} \| \\ \hom(C_\cdot^{cw}(X), R) \end{matrix} \right)$

<u>Proof.</u> Use UCT for $S^\cdot(X;R)$ & $C^\cdot_{cw}(X;R)$

\+ the map induced by i between the UCT SES's,

\+ the 5-lemma.

$\blacksquare$

We'll need a diag. approx. that works for $C_\cdot^{cw}(X)$ and is also compat. under i with a diag. approx. on $S_\cdot(X)$. Namely we need

$$\Delta : S_\cdot(X) \longrightarrow S_\cdot(X) \otimes S_\cdot(X), \quad \Delta^{cw} : C_\cdot^{cw}(X) \longrightarrow C_\cdot^{cw}(X) \otimes C_\cdot^{cw}(X) \quad \text{s.t.}$$

$$\begin{tikzcd} C_\cdot^{cw}(X) \arrow[r, "\Delta^{cw}"] \arrow[d, "i"'] & C_\cdot^{cw}(X) \otimes C_\cdot^{cw}(X) \arrow[d, "i \otimes i"] \\ S_\cdot(X) \arrow[r, "\Delta"] & S_\cdot(X) \otimes S_\cdot(X) \end{tikzcd} \qquad (*)$$

If we have this, then define ch. level $\cup$-products as:

$$\langle \varphi \cup \psi, \sigma \rangle := (\varphi \otimes \psi)(\Delta \sigma) \quad \forall \; \sigma : \Delta^n \longrightarrow X$$

$$\langle \varphi^{cw} \cup \psi^{cw}, \sigma \rangle := (\varphi^{cw} \otimes \psi^{cw})(\Delta^{cw} \sigma) \quad \forall \text{ n-cell } \sigma \text{ in } X.$$

Lecture #10A.

X = finite CW complex. $i: C_\cdot^{cw}(X) \longrightarrow S_\cdot(X)$ obvious inclusion.
Assume that:

1) $i: C_\cdot^{cw}(X) \longrightarrow S_\cdot(X)$ is a ch. map.

2) i induces an iso. in homology $i_*: H_*(C_\cdot^{cw}(X)) \xrightarrow{\cong} H_*(X)$.

3) $\exists\ \Delta: S_\cdot(X) \longrightarrow S_\cdot(X) \otimes S_\cdot(X)$, $\Delta^{cw}: C_\cdot^{cw}(X) \longrightarrow C_\cdot^{cw}(X) \otimes C_\cdot^{cw}(X)$ s.t.

$$\begin{tikzcd} C_\cdot^{cw}(X) \arrow[r, "\Delta^{cw}"] \arrow[d, "i"'] & C_\cdot^{cw}(X) \otimes C_\cdot^{cw}(X) \arrow[d, "i \otimes i"] \\ S_\cdot(X) \arrow[r, "\Delta"] & S_\cdot(X) \otimes S_\cdot(X) \end{tikzcd} \qquad (*)$$

Define ch. level $\cup$-products as:

$$\langle \varphi \cup \psi, \sigma \rangle := (\varphi \otimes \psi)(\Delta \sigma) \quad \forall \quad \sigma: \Delta^n \longrightarrow X$$

$$\langle \varphi^{cw} \cup \psi^{cw}, \sigma \rangle := (\varphi^{cw} \otimes \psi^{cw})(\Delta^{cw} \sigma) \quad \forall \text{ cellular cochains } \varphi^{cw}: C_p^{cw}(X) \longrightarrow R$$
$\psi^{cw}: C_q^{cw}(X) \longrightarrow R$, $p+q=n$,
and $\forall$ n-cell σ in X.

Note that $(*)$ implies that Δ^{cw} is a ch. map. This b.c. i & $i\otimes i$ are ch. maps + they are injective: so

$$\Delta^{cw}\circ\partial - \partial^{cw}_{\otimes}\circ\Delta^{cw} = 0 \iff (i\otimes i)\left(\Delta^{cw}\circ\partial - \partial^{cw}_{\otimes}\circ\Delta^{cw}\right) = 0.$$

$$\Rightarrow \quad i^*(\varphi\cup\psi) = i^*\varphi\cup i^*\psi, \text{ hence } \underbrace{i^*: H^*(X)\longrightarrow H^*(C^{\bullet}_{cw}(X)) \text{ is an iso.}}_{\text{of rings.}}$$

Here is an example when condit. 3 works.

3') Assume $\forall\ k\le\dim(X)$, $\forall\ p+q=k$ and $\forall$ k-dim. cell $f:\Delta^k\longrightarrow X$ of X, the $\underbrace{\text{p-front}}_{\text{face}}$ $f\rfloor_p:\Delta^p\longrightarrow X$ & the q-back face ${}_q\lfloor f:\Delta^q\longrightarrow X$ are both cells of X.

Note: $(3')\Rightarrow(3)$. Just take $\Delta: S.(X)\longrightarrow S.(X)\otimes S.(X)$ to be A-W Δ.

Define $\Delta^{cw}(\sigma) := \sum_{p+q=n} \sigma\rfloor_p\otimes {}_q\lfloor\sigma$ $\quad\forall$ n-cell σ of X.

by (3') these are cells.

Example. $X = \mathbb{T}^2$ 2-dim. torus.

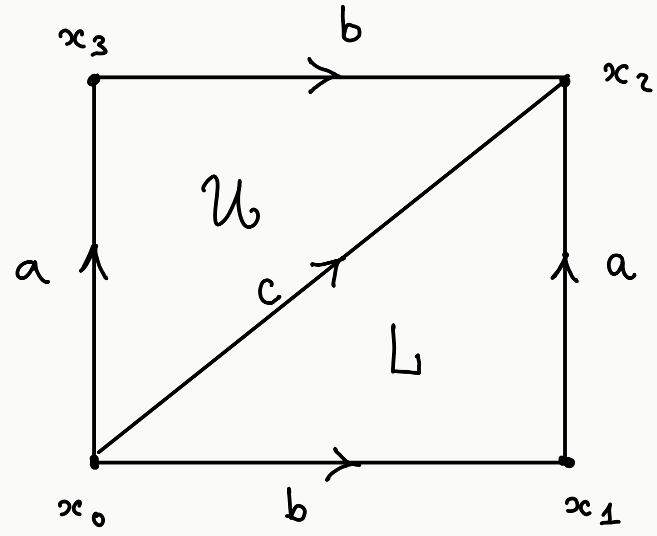
CW structure:

0-cells: $x = x_0 = x_1 = x_2 = x_3$.

1-cells: a, b, c

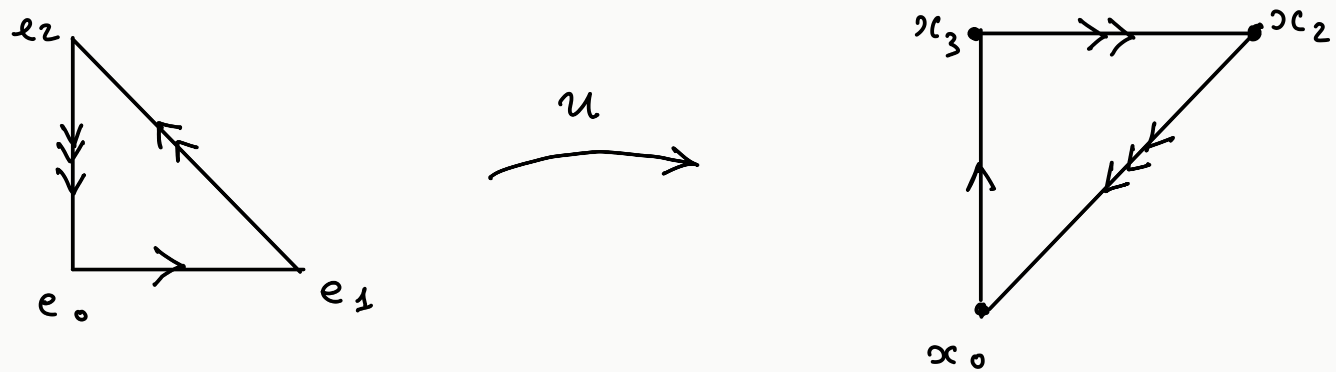
2-cells: $L = [x_0, x_1, x_2]$

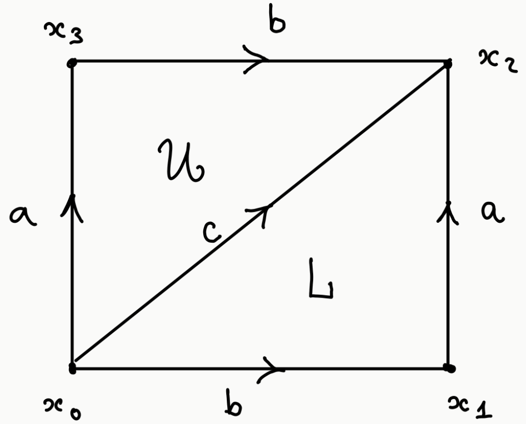
$U = [x_0, x_3, x_2]$



Note that, w.r.t. the picture, U is parametrized

as





CW structure:

0-cells: $x = x_0 = x_1 = x_2 = x_3$.

1-cells: a, b, c

2-cells: $L = [x_0, x_1, x_2]$

$\mathcal{U} = [x_0, x_3, x_2]$

$S_\cdot(X)$	$C_\cdot^{cw}(X)$
$\partial x = 0$	$\partial^{cw} x = 0$
$\partial a = \partial b = \partial c = 0$	$\partial^{cw}(a) = \partial^{cw}(b) = \partial^{cw}(c) = 0.$
$\partial L = a - c + b$	$\partial^{cw}(L) = b + a - c.$
$\partial \mathcal{U} = b - c + a$	$\partial^{cw}(\mathcal{U}) = a + b - c.$

$\Rightarrow$ i is a ch. map.

Assumption (3') holds.

Claim. i induces an iso. in homology.

proof. quite easy for H_0 & H_1.
For H_2: it's a bit more involved. $\Big\}$ exc.

$\partial x = 0$, $\partial a = \partial b = \partial c = 0$, $\partial \mathcal{U} = b - c + a$, $\partial L = a - c + b$. $\quad (\partial = \partial^{cw}$ here$)$

Write $x^*, a^*, b^*, c^*, \mathcal{U}^*, L^*$ for the dual basis to $x, a, b, c, \mathcal{U}, L$.

$(\langle a^*, a\rangle = 1$, $\langle a^*, b\rangle = \langle a^*, c\rangle = 0$ etc.$)$.

<u>$C_{cw}(X;R)$.</u> $\quad \delta(x^*) = -x^* \circ \partial = 0$. $\quad \langle \delta(a^*), \mathcal{U}\rangle = \langle a^*, b - c + a\rangle = 1$

$\langle \delta(a^*), L\rangle = 1$. $\Rightarrow$ $\delta(a^*) = \mathcal{U}^* + L^*$.

similarly, $\delta(b^*) = \mathcal{U}^* + L^*$, $\delta(c^*) = -(\mathcal{U}^* + L^*)$.

$\delta(\mathcal{U}^*) = \delta(L^*) = 0$.

$H^0_{cw} \cong R[x^*]$, $\quad H^1_{cw} \cong R\cdot[a^* + c^*] \oplus R[b^* + c^*]$, $\quad H^2_{cw} \cong R\cdot[\mathcal{U}^*] \cong R\cdot[-L^*]$.

Write $1 = [x^*]$, $\alpha = [a^* + c^*]$, $\beta = [b^* + c^*]$, $\mu = [\mathcal{U}^*] = -[L^*]$.

$\alpha \cup \beta = [\underbrace{(a^* + c^*)}_{\varphi} \cup \underbrace{(b^* + c^*)}_{\psi}]$. $\quad \langle \varphi \cup \psi, \mathcal{U}\rangle = \langle \varphi\cup\psi, [x_0, x_3, x_2]\rangle = -\langle \varphi, \overbrace{[x_0, x_3]}^{a}\rangle \cdot$

$\langle \psi, \underbrace{[x_3, x_2]}_{b}\rangle =$

$= -1 \cdot 1 = -1$.

$\langle \varphi \cup \psi, L\rangle = -\langle \varphi, b\rangle \cdot \langle \psi, a\rangle = 0$. $\Rightarrow$ $\varphi \cup \psi = -\mathcal{U}^*$.

$\alpha \cup \beta = [\varphi \cup \psi] = -\mu$.

Let's calculate $\Psi \cup \varphi$. $\langle \Psi \cup \varphi, \mathcal{U} \rangle = -\langle \Psi, a \rangle \cdot \langle \varphi, b \rangle = 0$.

$\langle \Psi \cup \varphi, L \rangle = -\langle \Psi, b \rangle \cdot \langle \varphi, a \rangle = -1$. $\Rightarrow$ $\Psi \cup \varphi = -L^*$. (Recall: $\varphi \cup \Psi = -\mathcal{U}^*$).

(So, on the chain level $\varphi \cup \Psi \neq -\Psi \cup \varphi$!)

$\beta \cup \alpha = -[L^*] = \mu$. (Recall $\alpha \cup \beta = -\mu$).

Exc. Show $\alpha \cup \alpha = \beta \cup \beta = 0$.
(holds in our case even if char$(R) = 2$.)

(Remark: $\alpha \cup \alpha = -\alpha \cup \alpha \Rightarrow 2\alpha \cup \alpha = 0$ and $2\beta \cup \beta = 0$.)

(This does NOT imply, in general, that $\alpha \cup \alpha = 0$, $\beta \cup \beta = 0$. It depends on R.)

$$H^*(\mathbb{T}^2; R) \cong R[\alpha, \beta] / \{\alpha^2 = 0, \beta^2 = 0, \alpha\beta = -\beta\alpha\}.$$

$$\overset{\Psi}{\mu} \longleftrightarrow \overset{\Psi}{\beta \cdot \alpha} .$$

Two more properties of $\cup$ product.

Recall:
$$H^p(X;R)\otimes_R H^q(X,A;R) \longrightarrow H^{p+q}(X,A;R)$$
$$H^p(X,A;R)\otimes_R H^q(X;R) \longrightarrow H^{p+q}(X,A;R)$$
$$H^p(X,A;R)\otimes_R H^q(X,A;R) \longrightarrow H^{p+q}(X,A;R)$$

Let $\delta^*: H^k(A;R) \longrightarrow H^{k+1}(X,A;R)$ be the connect. homo.

Let $i^*: H^*(X;R) \longrightarrow H^*(A;R)$ be the restrict. map (induced by the incl $i: A \to X$).

Then $\forall\ \alpha \in H^*(A;R)$, $\beta \in H^*(X;R)$ we have
$$\left.\begin{array}{l}\delta^*(\alpha \cup i^*\beta) = \delta^*\alpha \cup \beta \quad \& \quad \delta^*(i^*\beta \cup \alpha) = \beta \cup \delta^*(\alpha).\end{array}\right\}\text{exc.}$$

Denote by $j^*: H^*(X,A;R) \longrightarrow H^*(X;R)$ the map induced from $j: X \longrightarrow (X,A)$.
Then $\forall\ \alpha,\beta \in H^*(X,A;R)$ we have $j^*(\alpha\cup\beta) = j^*(\alpha)\cup j^*(\beta)$.

$\forall\ \alpha \in H^*(X;R)$, $\beta \in H^*(X,A;R)$ we have $j^*(\alpha\cup\beta) = \alpha\cup j^*(\beta)$ etc.

The cap product. Fix a ring R. X = space. Take both $S_\cdot(X)$ and $S^\cdot(X)$ with coeffs. in R. Let $\Delta : S_\cdot(X) \longrightarrow S_\cdot(X) \otimes S_\cdot(X)$ be a diag. approx. (over R). Define a map

$\boxed{\otimes = \otimes_R}$

$$S^p(X) \otimes S_n(X) \xrightarrow{\ \cap\ } S_{n-p}(X)$$
$$\varphi \otimes c \longmapsto \varphi \cap c := (\pi_q \otimes \varphi)\,\Delta c$$

we'll write $\varphi \cap c = (id \otimes \varphi)\,\Delta c$, using the convention that if $\psi \in S^i(X)$, $d \in S_j(X)$ then $\psi(d) = 0$ unless $i = j$.

$q := n-p$

$\pi_q : S_\cdot(X) \xrightarrow[\text{proj.}]{} S_q(X)$

Main example. If Δ = A-W diag. approx.

Let $\sigma : \Delta^{p+q} \longrightarrow X$ $\quad (p+q=n)$, $\varphi \in S^p(X)$ then $\varphi \cap \sigma = \ldots = (-1)^{pq}\varphi({}_p\lfloor\sigma)\cdot \sigma\rfloor_q$.

Prop. Let $\Delta = \Delta_{AW}$. Let $\varepsilon : S_\cdot(X) \longrightarrow R$ be the augment. Then:

1) $\varepsilon \cap c = c \quad \forall\ c \in S_\cdot(X)$.
2) $\forall\ \varphi \in S^p(X)$, $c \in S_p(X)$, $\quad \varepsilon(\varphi \cap c) = \varphi(c)$.
3) $\forall\ \varphi, \psi$ cochains $\quad (\varphi \cup \psi) \cap c = \varphi \cap (\psi \cap c)$.

<u>Prop.</u> Let $\cap$ be defined via a general Δ. Then

1) $\cap$ is natural w.r.t. maps in the sense that $\forall$ spaces X, Y, $X \xrightarrow{f} Y$ $\varphi \in S^p(Y)$, $c \in S_n(X)$ we have $f_c(f^c \varphi \cap c) = \varphi \cap f_c(c)$.

$$\begin{tikzcd}
S^p(X) \otimes S_n(X) \arrow[r, "\cap"] \arrow[d, shift left=6, "f_c"] & S_{n-p}(X) \arrow[d, "f_c"] \\
S^p(Y) \otimes S_n(Y) \arrow[u, shift left=6, "f^c"] \arrow[r, "\cap"] & S_{n-p}(Y)
\end{tikzcd}$$

Lecture #10B.

The cap product. Fix a ring R. $X=$ space. Take both $S_\cdot(X)$ and $S^\cdot(X)$ with coeffs. in R. Let $\Delta: S_\cdot(X) \longrightarrow S_\cdot(X)\otimes S_\cdot(X)$ be a diag. approx. (over R). Define a map

$$S^p(X)\otimes S_n(X) \xrightarrow{\ \cap\ } S_{n-p}(X)$$
$$\varphi\otimes c \longmapsto \varphi\cap c := (\pi_q\otimes\varphi)\Delta c$$

$\otimes = \otimes_R$

we'll write $\varphi\cap c = (id\otimes\varphi)\Delta c$, using the convention that if $\psi\in S^i(X)$, $d\in S_j(X)$ then $\psi(d)=0$ unless $i=j$.

$q:=n-p$

$\pi_q: S_\cdot(X) \xrightarrow[\text{proj.}]{} S_q(X)$

Main example. If $\Delta =$ A-W diag. approx.

Let $\sigma: \Delta^{p+q} \longrightarrow X$ $(p+q=n)$, $\varphi\in S^p(X)$ then $\varphi\cap\sigma = \ldots = (-1)^{pq}\varphi({}_p\lfloor\sigma)\cdot\sigma\rfloor_q$.

Prop. Let $\Delta = \Delta_{AW}$. Let $\varepsilon : S_\bullet(X) \longrightarrow R$ be the augment. Then:

1) $\varepsilon \frown c = c \quad \forall \; c \in S_\bullet(X)$.
2) $\forall\; \varphi \in S^p(X),\; c \in S_p(X),\; \varepsilon(\varphi \frown c) = \varphi(c)$.
3) $\forall\; \varphi, \psi$ cochains, c chain: $\quad (\varphi \smile \psi) \frown c = \varphi \frown (\psi \frown c)$.

Proof. 1 & 2: exc.

3) Let $\varphi \in S^p(X)$, $\psi \in S^q(X)$, $\sigma : \Delta^n \longrightarrow X$. $\quad n = p+q+r \quad (r := n-(p+q))$.

$$(\varphi\smile\psi)\frown\sigma = (-1)^{(p+q)\cdot r}\,(\varphi\smile\psi)\left({}_{p+q}\lfloor\sigma\right)\cdot \sigma\rfloor_r =$$

$$= \underbrace{(-1)^{(p+q)\cdot r}\,(-1)^{pq}}\;\varphi\Big(\underbrace{\left({}_{p+q}\lfloor\sigma\right)\rfloor_p}_{①}\Big)\cdot\psi\Big(\underbrace{{}_q\lfloor\left({}_{p+q}\lfloor\sigma\right)}_{②}\Big)\cdot\underbrace{\sigma\rfloor_r}_{③}$$

$$\varphi\frown(\psi\frown\sigma) = (-1)^{q\cdot(p+r)}\,\varphi\frown\left(\psi\left({}_q\lfloor\sigma\right)\cdot\sigma\rfloor_{p+r}\right) = \underbrace{(-1)}^{q(p+r)}\,\psi\Big(\underbrace{{}_q\lfloor\sigma}_{②'}\Big)\cdot\underbrace{(-1)}^{p\cdot r}\Big(\underbrace{\left(\sigma\rfloor_{p+r}\right)\rfloor_r}_{③'}\Big)\cdot\varphi\Big(\underbrace{{}_p\lfloor\left(\sigma\rfloor_{p+r}\right)}_{①'}\Big)$$

Now: $(-1)^{(p+q)\cdot r + p\cdot q} = (-1)^{q\cdot(p+r)+pr}$, so the signs agree.

$③' = \left(\sigma\rfloor_{p+r}\right)\rfloor_r = \sigma\rfloor_r = ③$. $\quad ② = {}_q\lfloor\left({}_{p+q}\lfloor\sigma\right) = {}_q\lfloor\sigma = ②'$. $\quad ① = \left({}_{p+q}\lfloor\sigma\right)\rfloor_p = {}_p\lfloor\left(\sigma\rfloor_{p+r}\right) = ①'$.

<u>Prop.</u> Let $\cap$ be defined via a general Δ. Then

1) $\cap$ is natural w.r.t. maps in the sense that $\forall$ spaces X, Y, $X \xrightarrow{f} Y$, $\varphi \in S^p(Y)$, $c \in S_n(X)$ we have $f_c(f^c\varphi \cap c) = \varphi \cap f_c(c)$.

$$\begin{tikzcd} S^p(X) \otimes S_n(X) \arrow[r, "\cap"] \arrow[d, "f_c"] & S_{n-p}(X) \ni f^c\varphi \cap c \arrow[d, "f_c"] \\ S^p(Y) \otimes S_n(Y) \arrow[u, "f^c"] \arrow[r, "\cap"] & S_{n-p}(Y) \ni \varphi \cap f_c(c) \end{tikzcd}$$

$c \in S_n(X)$, $\varphi \in S^p(Y)$

2) $\cap$ is a chain map in the following sense:

$\forall\ \varphi \in S^p(X),\ c \in S_n(X)$ we have $\partial(\varphi \cap c) = \delta\varphi \cap c + (-1)^{|\varphi|}\varphi \cap \partial c$.

<u>Proof.</u> 1) exc. Follows from naturality of Δ.

2) Put $p := |\varphi|$. Consider

$$\partial\otimes id \longleftrightarrow \partial$$

$$S_\bullet(X)\otimes_R S_p(X) \xrightarrow{id\otimes\varphi} S_\bullet(X)\otimes_R R \cong S_m(X)$$

This is "almost" a ch. map if we endow $S_\bullet(X)\otimes S_p(X)$ with $\partial\otimes id$.
The only problem is the sign:

$$(id\otimes\varphi)\circ(\partial\otimes id) = (-1)^{|\varphi|}\partial\otimes\varphi = (-1)^{|\varphi|}(\partial\otimes id_R)\circ(id\otimes\varphi). \quad (**)$$

Let $\varphi\in S^p(X)$, $c\in S_n(X)$.

$$\partial(\varphi\cap c) = \partial((id\otimes\varphi)\Delta c) = (\partial\otimes id_R)\circ(id\otimes\varphi)\Delta c. \quad (*)$$

$$\partial\otimes id_R \longleftrightarrow \partial$$

$$\begin{array}{ccccccc} S_n(X) & \xrightarrow{\Delta} & \bigoplus_{r+l=n} S_r(X)\otimes_R S_l(X) & \xrightarrow{id\otimes\varphi} & S_{n-p}(X)\otimes_R R & \cong & S_{n-p}(X) \\ \Psi & & \Psi & & & & \\ c & & \Delta c & & (id\otimes\varphi)\Delta c & & \end{array}$$

We've seen: $(*) = (-1)^{|\varphi|}(id\otimes\varphi)\circ(\partial\otimes id)\Delta c = (-1)^{|\varphi|}(id\otimes\varphi)\circ(\partial_\otimes - id\otimes\partial)\Delta c$

$(**)$

$= (-1)^{|\varphi|}(id\otimes\varphi)\Delta(\partial c) + (-1)^{|\varphi|+1}(id\otimes\varphi)\circ(id\otimes\partial)\Delta c =$

$\partial_\otimes\Delta = \Delta\partial$

$$= (-1)^{|\varphi|}\varphi \cap \partial c + (id \otimes \delta\varphi)\Delta c = (-1)^{|\varphi|}\varphi\cap\partial c + (\delta\varphi)\cap c.$$

$\boxed{/\!/\!/}$

Cor. The chain-level cap prod. descends to homology:

$$H^p(X) \otimes H_n(X) \xrightarrow{\cap} H_{n-p}(X)$$
$$\alpha \otimes a \longmapsto \alpha \cap a,$$

which is independent of the particular choice of Δ. Moreover:

1) $1 \cap a = a \quad \forall \; a \in H_*(X)$.

2) If $\alpha \in H^p(X)$, $a \in H_p(X) \Rightarrow \varepsilon_*(\alpha \cap a) = \langle \alpha, a \rangle$.

kronecker pairing.

3) $(\alpha \cup \beta) \cap a = \alpha \cap (\beta \cap a) \quad \forall \; \alpha, \beta \in H^*(X)$, $a \in H_*(X)$.

4) If $f : X \longrightarrow Y$, $\alpha \in H^*(Y)$, $a \in H_*(X)$, then $f_*(f^*\alpha \cap a) = \alpha \cap f_* a$.

Cor. Denote $\langle \cdot , \cdot \rangle : H^p(X;R) \otimes_R H_p(X;R) \longrightarrow R$ be the Kronecker pairing.

1) Let $\alpha, \beta \in H^*(X;R)$ of pure deg. s.t. $|\alpha| + |\beta| = p$, and let $c \in H_p(X;R)$.

Then $\langle \alpha \cup \beta , c \rangle = \langle \alpha , \beta \cap c \rangle \in R$.

2) Let $f : X \longrightarrow Y$, $\alpha \in H^p(Y)$, $c \in H_p(X)$. Then $\langle f^* \alpha , c \rangle = \langle \alpha , f_* c \rangle$.

Proof. 1) $\langle \alpha \cup \beta , c \rangle = \varepsilon_* \big((\alpha \cup \beta) \cap c \big) = \varepsilon_* \big(\alpha \cap (\beta \cap c) \big) = \langle \alpha , \beta \cap c \rangle$.

2) $\langle \alpha , f_* c \rangle = \varepsilon^X_* \big(\alpha \cap f_* c \big) = \varepsilon^Y_* \big(f_* (f^* \alpha \cap c) \big) = \varepsilon^X_* \big(f^* \alpha \cap c \big) = \langle f^* \alpha , c \rangle$.

$$(\varepsilon^Y_* \circ f_* = \varepsilon^X_*)$$

$\blacksquare$

<u>Relative versions</u> X = space, $A \subset X$ subspace. R = ring.

If $c \in S_n(A)$, $\varphi \in S^p(X)$. $\Rightarrow$ $\varphi \cap c$ is a chain in A in $S_{n-p}(A)$.

b.c. naturality of Δ.

$$\Rightarrow \cap : S^p(X) \otimes \underset{\substack{\shortparallel \\ S_n(X)/S_n(A)}}{S_n(X,A)} \longrightarrow \underset{\substack{\shortparallel \\ S_{n-p}(X)/S_{n-p}(A)}}{S_{n-p}(X,A)} .$$

$$\Rightarrow \cap : H^p(X) \otimes H_n(X,A) \longrightarrow H_{n-p}(X,A).$$

Let $\varphi \in S^p(X,A)$, i.e. $\varphi : S_p(X)/S_p(A) \longrightarrow R$. Let $c \in S_n(X)$.

Put $\hat{\varphi} = \left(S_p(X) \xrightarrow{pr} S_p(X)/S_p(A) \xrightarrow{\varphi} R \right)$ (i.e. $\hat{\varphi} = j^c \varphi$, where $j^c : S^p(X,A) \to S^p(X)$)

Consider $\hat{\varphi} \cap c$. Note that if $c' = c + a$ with $a \in S_n(A)$ then

$\hat{\varphi} \cap c' = \hat{\varphi} \cap c$ $\left(\text{b.c. } \hat{\varphi}\big|_{S_p(A)} \equiv 0 \right)$. $\Rightarrow$ the map $\varphi \otimes c \longmapsto \hat{\varphi} \cap c$

induces a well defined "chain map" $\cap : S^p(X,A) \otimes S_n(X,A) \longrightarrow S_{n-p}(X)$.

In homology: $\cap : H^p(X,A) \otimes H_n(X,A) \longrightarrow H_{n-p}(X)$.

Let $A, B \subset X$.

Claim. The chain level cap product induces a map

$$S^p(X,A) \otimes S_n(X)/S_n^{A,B}(X) \xrightarrow{\ \cap\ } S_{n-p}(X)/S_{n-p}(B).$$

Proof. Let $\varphi \in S^p(X,A)$ viewed as $\varphi : S_p(X) \longrightarrow R$ with $\varphi|_{S_p(A)} \equiv 0$.

Let $c \in S_n(X)$ and $\lambda = \lambda^A + \lambda^B \in S_n^{A,B}(X)$ with $\lambda^A \in S_n(A)$, $\lambda^B \in S_n(B)$.

$$\varphi \cap (c+\lambda) = \varphi \cap c + \underbrace{\varphi \cap \lambda^A}_{0} + \underbrace{\varphi \cap \lambda^B}_{S_{n-p}(B)} \implies \varphi \cap c \ \&\ \varphi \cap (c+\lambda) \text{ differ by an element of } S_{n-p}(B).$$

$\boxed{/\!/}$

In cohomology we get: $H^p(X,A) \otimes H_n\left(S_\cdot(X)/S_\cdot^{A,B}(X)\right) \xrightarrow{\ \cap\ } H_{n-p}(X,B)$.

Conclusion: If $S_\cdot^{A,B}(X) \longrightarrow S_\cdot(A \cup B)$ is a quasi-iso. (induces iso. in homology)

then $\cap$ gives $H^p(X,A) \otimes H_n(X, A\cup B) \xrightarrow{\ \cap\ } H_{n-p}(X,B)$.

This happens e.g. when $A, B \subset X$ are open, or when either A or B is ϕ,
or when $A \subset B$.

Some examples of cohomology rings.

1) $H^*(\mathbb{T}^2; R) \cong R[\alpha, \beta] \big/ \{\alpha\cdot\alpha = 0,\ \beta\cdot\beta = 0,\ \alpha\cdot\beta = -\beta\cdot\alpha\}$. $(|\alpha| = |\beta| = 1.)$

2) $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha] \big/ \{\alpha^{n+1} = 0\}$.

3) $H^*(\mathbb{C}P^n; R) \cong R[x] \big/ \{x^{n+1} = 0\}$, $|x| = 2$.

If $x_{2j} \in \underset{R\cdot x_{2j}}{\underset{\shortparallel}{H^{2j}(\mathbb{C}P^n; R)}} \cong R$ is a generator

Alternatively, denote by $\alpha_j \in \underset{\mathbb{Z}_2}{\underset{\shortparallel}{H^j(\mathbb{R}P^n; \mathbb{Z}_2)}}$ the generator, $\forall\ 0 \leq j \leq n$.

Then $\alpha_i \cup \alpha_j = \alpha_{i+j}$ $\forall\ 0 \leq i, j$ s.t. $i + j \leq n$.

then $x_{2i} \cup x_{2j} = c_{ij} \cdot x_{2(i+j)}$

for some $0 \neq c_{ij} \in R$. In fact, it is possible to choose generators

$x_{2j} \in H^{2j}(\mathbb{C}P^n; R)$ s.t. $c_{ij} = 1$ $\forall\ i, j$, i.e. $x_{2i} \cup x_{2j} = x_{2(i+j)}$.

Lecture #11A.

1) $R = \mathbb{Z}_2$, $\quad H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha] / \{\alpha^{n+1} = 0\}$, $\quad |\alpha| = 1$.

$H^k(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2 \cdot (\underbrace{\alpha \cup \ldots \cup \alpha}_{\times k})$, where $\alpha \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the generator

$\forall \quad 0 \leq k \leq n$.

2) $H^*(\mathbb{C}P^n; R) \cong R[x] / \{x^{n+1} = 0\}$, $\quad |x| = 2$. $\quad$ (Assume $n \geq 1$)

$\exists \quad x \in H^2(\mathbb{C}P^n; R)$ s.t. $H^2(\mathbb{C}P^n; R) = R \cdot x$ and

$H^{2k}(\mathbb{C}P^n; R) = R \cdot (\underbrace{x \cup \ldots \cup x}_{\times k})$. $\quad H^l(\mathbb{C}P^n; R) = 0 \quad \forall \quad l = \text{odd}$.

$0 \leq k \leq n$

3) $\mathbb{T}^n := \underbrace{S^1 \times \ldots \times S^1}_{\times n}$. $\quad H^*(\mathbb{T}^n; R) \cong \Lambda_R[\alpha_1, \ldots, \alpha_n]$ exterior algebra on $\alpha_1, \ldots, \alpha_n$

Write $A := \Lambda_R[\alpha_1, \ldots, \alpha_n]$, $\quad A = A^0 \oplus A^1 \oplus \ldots \oplus A^n$. $\quad |\alpha_1| = \ldots = |\alpha_n| = 1$.

A^k = a free R-module generated by $\alpha_{i_1} \cdot \ldots \cdot \alpha_{i_k}$, $\quad 1 \leq i_1 < \ldots < i_k \leq n$

We'll add the following relations: $\alpha_i \cdot \alpha_j = -\alpha_j \alpha_i$, $\alpha_i \cdot \alpha_i = 0$, $\forall\ i,j$.

This defines a product on $A = \Lambda_R[\alpha_1, \dots, \alpha_n]$.

An application the cup products

Thm. X = space, $X = \mathcal{U} \cup V$ with $\mathcal{U}$ & V open and acyclic subsets of X.

Then $\forall\ \alpha, \beta \in H^*(X;R)$ with $|\alpha|, |\beta| > 0$ we have $\alpha \cup \beta = 0$.

Cor. None of $\mathbb{T}^n, \mathbb{R}P^n$, $n \geq 2$, as well $\mathbb{C}P^n$, $n \geq 2$, can be written as a union of two open acyclic subsets.

Remark. $\mathbb{T}^1 = S^1$, $\mathbb{R}P^1 \approx S^1$, $\mathbb{C}P^1 \approx S^2$ and more generally S^n can be written as a union of two contractible subsets.

Proof of thm. Let $\alpha \in H^p(X)$, $\beta \in H^q(X)$ with $p,q \geq 1$.

We have the LES of $(X,\mathcal{U})$: $\cdots \longrightarrow H^p(X,\mathcal{U}) \xrightarrow{j_{\mathcal{U}}^*} H^p(X) \xrightarrow{i_{\mathcal{U}}^*} H^p(\mathcal{U}) \to \cdots$

$\mathcal{U}$ is acyclic $\Rightarrow$ ~~[illegible]~~ $i_{\mathcal{U}}^*(\alpha) = 0$. $\Rightarrow$ $\alpha = j_{\mathcal{U}}^*(\alpha')$ for some $\alpha' \in H^p(X,\mathcal{U})$.

Similarly by considering (X,V), we have $\beta = j_V^*(\beta')$ — " — $\beta' \in H^q(X,V)$.

Since $\mathcal{U}$ & V are open we have a version of the cup product as follows:

$$H^p(X,\mathcal{U}) \otimes H^q(X,V) \xrightarrow{\ \cup\ } \underbrace{H^{p+q}(X,\mathcal{U}\cup V)}_{H^{p+q}(X,X)\,=\,0.}$$

$\Rightarrow$ $\alpha' \cup \beta' = 0$.

Claim. The map $j'^*: H^{p+q}(X,\mathcal{U}\cup V) \longrightarrow H^{p+q}(X)$ maps $\alpha' \cup \beta'$ to $\alpha \cup \beta$.

From the claim it follows that $\alpha \cup \beta = 0$.

Proof of the claim.

If $\mathcal{U}, V \subset X$ are open then $\exists$ a commut. diag.

$$
\begin{tikzcd}
H^p(X,\mathcal{U}) \otimes H^q(X,V) \arrow[r, "\cup"] \arrow[d, "j_{\mathcal{U}}^* \otimes j_V^*"'] \arrow[dr, phantom, "c"] & H^{p+q}(X, \mathcal{U} \cup V) \arrow[r, "j'^*"] & H^{p+q}(X) \\
H^p(X) \otimes H^q(X) \arrow[urr, "\cup"'] & &
\end{tikzcd}
$$

Indeed:

$$
\begin{tikzcd}
& H^{p+q}\left(\left(S_\cdot(X)/S_\cdot^{\mathcal{U},V}(X)\right)^*\right) \arrow[d, "\cong"'] \arrow[dr] \arrow[d, phantom, shift left=8, "c"] & \\
H^p(X,\mathcal{U}) \otimes H^q(X,V) \arrow[ur, "\cup"] \arrow[r, "\cup"] \arrow[d, "j_{\mathcal{U}}^* \otimes j_V^*"'] & H^{p+q}(X, \mathcal{U} \cup V) \arrow[r, "j'^*"] & H^{p+q}(X) \\
H^p(X) \otimes H^q(X) \arrow[urr, "\cup"'] & &
\end{tikzcd}
$$

This triangle is obviously commut.

The "outer" diag. commutes (b.c. it commutes on the ch. level) $\Rightarrow$ original diag. commutes. $\blacksquare$

Manifolds & Poincaré duality.

Def. A (topological) manifold of dim n is a top. space M s.t.

1) M is Haus.

2) $\forall\, x \in M$, $\exists$ a nbhd. $\mathcal{U}_x \subset M$ of x and a homeo. $\varphi_x : \mathbb{R}^n \xrightarrow{\approx} \mathcal{U}_x$

(w.l.o.g. we may assume $\varphi_x(0) = x$.)

called a chart around x.

Examples.

1) $M = \mathbb{R}^n$, or $M =$ open subset in $\mathbb{R}^n$ is a manifold.

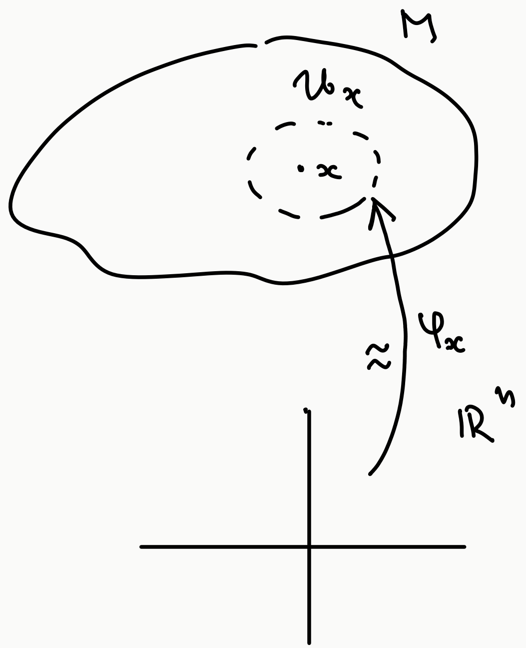
(Note: $\operatorname{Int} B^n(\varepsilon) \approx \mathbb{R}^n$).

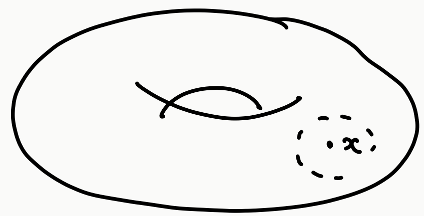
2) $M = S^n$. Cover S^n by $2n+2$ charts

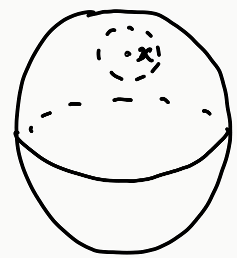
$$\mathcal{U}_i^+ = \{(x_1, \dots, x_{n+1}) \in S^n : x_i > 0\}$$

$$\mathcal{U}_i^- = \{(x_1, \dots, x_{n+1}) \in S^n : x_i < 0\}$$

$i = 1, \dots, n+1$. $\quad \mathcal{U}_i^{\pm} \approx \operatorname{Int} B^n(1) \approx \mathbb{R}^n$.





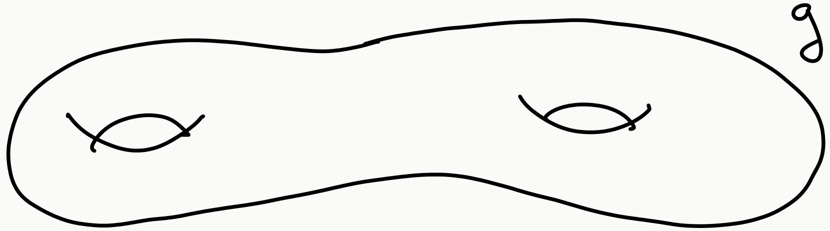


3) $\mathbb{RP}^n = S^n/\sim$. $\mathbb{RP}^n$ is locally homeo. to S^n. $\Rightarrow$ $\mathbb{RP}^n$ is an n-dim. manif.
($x \sim -x \ \forall x \in S^n$)

4) ~~[illegible]~~ $M' \xrightarrow{\pi} M$ covering. M' is an n-dim. manif.
iff M is an n-dim manif. (exc.)

5) M_1 = n_1-dim. manif. , M_2 = n_2-dim. manif. $\Longrightarrow$ $M_1 \times M_2$ is an (n_1+n_2)-dim manif. (so $\mathbb{T}^n = S^1 \times \ldots \times S^1$ is an n-dim. manif.).

6) Σ_g = surface of genus g is a 2-dim. manif. ($\tilde{\Sigma}_g = \mathbb{H} \approx \mathbb{R}^2$)
$g \geq 2$ upper half space


$g=2$

$$\begin{pmatrix} \Sigma_1 = \mathbb{T}^2 \\ \Sigma_0 = S^2 \end{pmatrix}$$

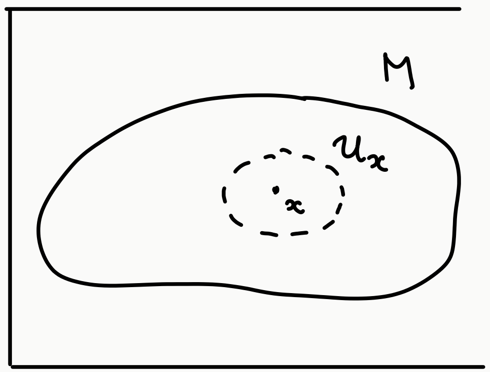
7) $M = \mathbb{C}P^n$ is a $2n$-dim. manif.

Cover $\mathbb{C}P^n$ by charts $\mathcal{U}_i$, $\quad \mathcal{U}_i := \{[z_0 : \dots : z_n] : z_i \neq 0\} \subset \mathbb{C}P^n$

$$\mathcal{U}_i \longrightarrow \mathbb{C}^n \approx \mathbb{R}^{2n} \qquad [z_0 : \dots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right)$$

Let M be an n-manifold. Let $x \in M$, $\varphi : \mathbb{R}^n \longrightarrow \mathcal{U}_x$ a chart around x.

$$H_i(M, M \setminus \{x\}) \underset{\text{excis.}}{\cong} H_i\big(M \setminus (M \setminus \mathcal{U}_x), M \setminus \{x\} \setminus (M \setminus \mathcal{U}_x)\big) = H_i(\mathcal{U}_x, \mathcal{U}_x \setminus \{x\}) \cong$$



$$\underset{\text{using } \varphi}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \underset{\text{b.c. } \tilde{H}_i(\mathbb{R}^n) = 0.}{\cong} \tilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{i-1}(S^{n-1})$$

$\Rightarrow H_i(M, M \setminus \{x\}) = 0 \quad \forall\, i \neq n$ and $H_n(M, M \setminus \{x\}) =$ infinite cyclic group $\cong \mathbb{Z}$.

We call $H_i(M, M \setminus \{x\})$ the local homology of M at x.

<u>Rem.</u> The chart $\varphi : \mathbb{R}^n \longrightarrow \mathcal{U}_x$ in fact gives us an iso. $\rho_\varphi : H_n(M, M \setminus \{x\}) \longrightarrow \tilde{H}_{n-1}(S^{n-1})$
$\cong \mathbb{Z}$.

Exc. Show that if $\varphi' = \varphi \circ \tau$ is another chart, where $\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear iso., then $\rho_{\varphi'} = \varepsilon_\tau \cdot \rho_\varphi$, where $\varepsilon_\tau = +1$ if τ is orientat. preserv. (i.e. $\det \tau > 0$) and $\varepsilon_\tau = -1$ is τ is orient. reversing $(\det \tau < 0)$.

Def. A local orientation of M at x is a choice of a generator $\mu_x \in \underbrace{H_n(M, M \setminus \{x\})}_{\text{infinite cyclic grp.}}$. $\exists$ exactly two possible loc. orient. μ_x & $-\mu_x$.

Rem. If $U_x \subset M$ is (the image of) a chart (fix $\varphi_x : \mathbb{R}^n \xrightarrow{\cong} U_x$), then μ_x induces local orient. μ_y for all $y \in U_x$. Indeed let $y \in U_x$, and let $B_0 \subset \mathbb{R}^n$ be a ball that contains both $\varphi_x^{-1}(x)$ & $\varphi_x^{-1}(y)$. Put $B := \varphi_x(B_0) \subset U_x$.

Then: $H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_x^{-1}(x)) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_0) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi_x^{-1}(y)) \cong$

$\cong H_n(M, M \setminus \{y\})$. The compos. of these iso's gives us an iso. $H_n(M, M \setminus \{x\}) \xrightarrow{\cong} H_n(M, M \setminus \{y\})$ which is canonical (independent of φ).

<u>Notation.</u> $A \subset M$ subset. We'll write $H_i(M|A;G) := H_i(M, M \setminus A; G)$

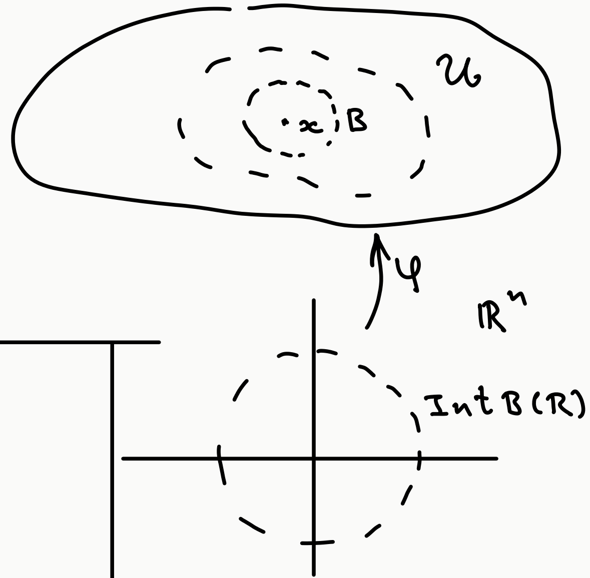
We call this the local homology of M at A. (For $G = \mathbb{Z}$ we omit G from notat.)

Lecture #11B.

M = n-dim. manifold, $A \subset M$ subset. $H_i(M|A;G) := H_i(M, M \setminus A; G)$
local homology of M at A.

Ball charts

$$\begin{array}{ccccc} x \in & B & \subset & \mathcal{U} \subset M^n \\ & \varphi|_{\mathrm{Int}\,B(R)} \uparrow \approx & & \approx \uparrow \varphi \\ & \mathrm{Int}\,B(R) & \subset & \mathbb{R}^n \end{array}$$



Let $B \subset M$ be a ball chart

$$H_n(M|B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B(R)) \cong \mathbb{Z}.$$

depends on φ

$\forall\ y \in B$ we have

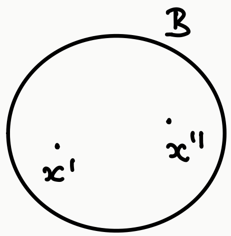
$$\begin{array}{ccc} H_n(M, M \setminus B) & \xrightarrow[\cong]{inc_*} & H_n(M, M \setminus \{y\}) \\ \| & & \\ H_n(M|B) & \xrightarrow[=]{L_y} & H_n(M|y) \end{array}$$

canonical iso.

$\forall\ x', x'' \in B$ we get

$$\begin{array}{ccc} H_n(M|x') & \xrightarrow{\cong} & H_n(M|x'') \\ L_{x'} \cong \nwarrow & & \nearrow \cong L_{x''} \\ & H_n(M|B) & \end{array}$$

canonical



<u>Def.</u> Let M be an n-manif. An orientation of M is a funct. $M \ni x \longmapsto \mu_x$

with $\mu_x \in H_n(M|x)$, that assigns $\forall\, x \in M$ a loc. orient. μ_x s.t.

$\forall\, x \in M$ $\exists$ a chart $\mathcal{U}$ around x and a ball chart $B \subset \mathcal{U}$ s.t.

$L_y L_x^{-1}(\mu_x) = \mu_y$

$\forall\, y \in B.$

$$\begin{tikzcd} & H_n(M|B) \arrow[dl, "L_x"', "\cong"] \arrow[dr, "\cong"', "L_y"] & \\ H_n(M|x) & & H_n(M|y) \end{tikzcd}$$

or, in other words, $\exists$ $\mu_B \in H_n(M|B)$ a generator, s.t.

$L_y(\mu_B) = \mu_y \quad \forall\, y \in B.$

If an orient. on M exists we say M is orientable.
When we fix an orientation, we say M is oriented.

A useful 2-sheet covering of M.

> convention: we do NOT require that a covering $\overbrace{\text{space}}^{X}$, $X \to Y$, is connected

Let M be an n-manif.

$\tilde{M} := \{ (x, \mu_x) : x \in M, \mu_x$ is a loc. orient. of M at x, i.e. $\mu_x \in H_n(M|x)$ is a generator $\}$.

$p : \tilde{M} \longrightarrow M$, $\quad p(x, \mu_x) := x$, $\quad$ 2:1 map. $\quad p^{-1}(x) = \{ (x, \mu_x), (x, -\mu_x) \}$.

Top. on $\tilde{M}$. Let $B \subset \mathcal{U} \subset M$ be a chart & a ball chart.

Let $\mu_B \in H_n(M|B)$ be a generator. $\forall\, x \in B$, we have an iso. $H_n(M|B) \xrightarrow[\cong]{L_x} H_n(M|x)$

Put $\mathcal{W}(\mu_B) := \{ (x, \mu_x) : x \in B, \mu_x = L_x(\mu_B) \}$.

The sets $\{\mathcal{W}(\mu_B)\}_{\mathcal{U}, \mu_B}$ form the basis of a topology on $\tilde{M}$. (exc.)

Moreover $p : \tilde{M} \longrightarrow M$ sends $\mathcal{W}(\mu_B)$ homeomorphically onto $\mathcal{U}$.

Conclusion. $\tilde{M}$ is an n-manif. and $\tilde{p}$ is a 2:1 covering.

Moreover $\tilde{M}$ is orientable. Indeed, an or. on $\tilde{M}$ is given by

$$(x,\mu_x) \longmapsto \tilde{\mu} \in H_n(\tilde{M} \mid (x,\mu_x)) \cong H_n(W(\mu_B) \mid (x,\mu_x)) \cong H_n(B \mid x) \cong H_n(M \mid x) \ni \mu_x$$

where $\tilde{\mu}$ corresponds to μ_x under the above iso.

Actually $\tilde{M}$ is canonically oriented.

<u>Thm.</u> Assume M is a connected n-manif. Then $\tilde{M}$ has at most two connected components. Moreover, M is orientable iff $\tilde{M}$ has <u>two</u> connected components. In particular, if M is simply connected or more generally if $\pi_1(M)$ has no subgroup of index 2, then M is orientable.

For the proof, we need the following

<u>Lemma.</u> Let $p: X \to Y$ be a 2:1 covering, with Y path-connected. Then:

1) X is path-connected iff $\exists$ a loop γ in Y that lifts to a non-closed path in X.
2) X can have at most two path-connected components. When it has two i.e. $X = X' \sqcup X''$, then $p|_{X'}: X' \to Y$, $p|_{X''}: X'' \to Y$ are homeomorphisms.

Proof of Thm. Assume M is orientable. $\Rightarrow$ $\exists$ an embedd.

$$\begin{tikzcd} \tilde{M} \arrow[d, "p"'] \\ M \arrow[u, bend right, "j"'] \end{tikzcd}$$

$j : M \hookrightarrow \tilde{M}$ coming from a choice of an orient. $j(x) = (x, \mu_x)$.

and $j'(x) := (x, -\mu_x)$, $j' : M \hookrightarrow \tilde{M}$. Clearly j' is also an embedd.

Also $\operatorname{image} j \cap \operatorname{image} j' = \phi$ $\Rightarrow$ $\tilde{M} = j(M) \sqcup j'(M)$.

Conversely, suppose $\tilde{M}$ is disconnected, $\tilde{M} = C_1 \sqcup C_2$. By the lemma

$p|_{C_1} : C_1 \longrightarrow M$ is a homeo. and we obtain an orient. on M.

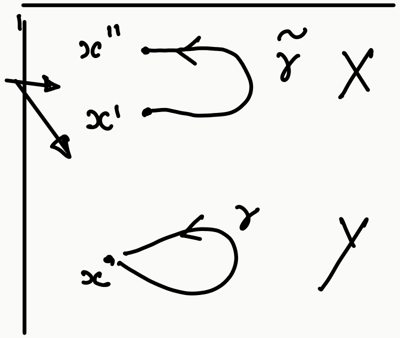
Now, if $\pi_1(M)$ has no subgroups of index 2 $\Rightarrow$ any covering 2:1

$X \to M$ is disconnected (b.c. path-connected coverings d:1 are in 1-1 corresp. with subgroups of index d of $\pi_1(M)$).

$\blacksquare$

Proof of the lemma. Let $p : X \to Y$ be a 2:1 covering. If X is path connected

then obviously $\exists$ a loop γ in Y which doesn't lift to a loop.



Conversely, suppose $\gamma : I \longrightarrow Y$ is a loop with with $\gamma(0) = \gamma(1) = x_0 \in Y$, and $\tilde{\gamma}$ is a lift of γ with $\tilde{\gamma}(0) = x_0'$, $\tilde{\gamma}(1) = x_0''$, $x_0' \neq x_0''$.

Now let $\tilde{x} \in X$ be any point. Put $x := p(\tilde{x})$. Y is path-connected, so take a path α in Y ~~from~~ with $\alpha(0) = x$, $\alpha(1) = x_0$.

By lifting α starting at $\tilde{x}$ we get a path from $\tilde{x}$ to one of x_0' or x_0''. But x_0' & x_0'' are in the same path-connected comp. of X.

$\Rightarrow$ $\tilde{x}$ is also in that component. This proves statement 1.

2) Suppose X is not path-connected. Let X' be a path connected comp. of X. Obviously $\forall$ 2 points $x_1, x_2 \in X'$ with $x_1 \neq x_2$ we have $p(x_1) \neq p(x_2)$ otherwise we'll have a non-closed path in X' which projects under p to a loop in Y. Contradiction, by 1.

Also $p(X') = Y$, because given $y \in Y$ just choose $x_0' \in X'$, put $x_0 := p(x_0')$, take a path $\gamma : I \longrightarrow Y$ with $\gamma(0) = x_0$ & $\gamma(1) = y$ and now lift γ to a path $\tilde{\gamma} : I \longrightarrow X$ with $\gamma(0) = x_0'$. Then $\tilde{\gamma}(1) \in X'$ & $p(\tilde{\gamma}(1)) = y$.

So, $p : X' \longrightarrow Y$ is 1-1. By the def. of covering spaces, p is a loc. homeo.

$\Rightarrow$ p is a homeo. The fact that $\# \pi_0(X) = 2$ is straightforward. $\blacksquare$

A more general covering.

Define $\tilde{M}_{\mathbb{Z}} = \{ (x, \alpha_x) : \cancel{\forall x}\; x \in M,\ \alpha_x \in H_n(M|x) \}$ (α_x is not necess. a generator.)

$p : \tilde{M}_{\mathbb{Z}} \longrightarrow M \qquad p(x, \alpha_x) := x.$

Top on $\tilde{M}_{\mathbb{Z}}$. Let $B \subset M$ be a ball chart

$$\mathcal{W}(\alpha_B) = \{ (x, \alpha_x) \in \tilde{M}_{\mathbb{Z}} : x \in B,\ \exists\ \alpha_B \in H_n(M|B) \text{ s.t. } L_x(\alpha_B) = \alpha_x \ \forall x \in B \}.$$

basis for a top. on $\tilde{M}$. Inside $\tilde{M}_{\mathbb{Z}}$ we have $M_0 \approx M$, $M_0 = \{ (x, 0) : x \in M \}$

$M_k := \{ (x, \alpha_x) : \cancel{\forall x}\; x \in M,\ \alpha_x$ is k times a generator of $H_n(M|x) \}$

$1 \leq k \in \mathbb{Z}$

<u>Def.</u> Let $X \xrightarrow{p} Y$ be a covering. A section $s : Y \longrightarrow X$ is a (<u>contin.</u>) map

$s : Y \longrightarrow X$ s.t. $p \circ s = id_X$

$$\begin{tikzcd} X \arrow[d, "p"'] \\ Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

So, an orientation on M is a <u>section</u> $\mu : M \longrightarrow \tilde{M}$.

or, a section $\alpha : M \longrightarrow \tilde{M}_{\mathbb{Z}}$ with $\alpha_x \in H_n(M|x)$ a generator $\forall x$.
$$x \longmapsto \alpha_x$$

<u>A further generaliz.</u> Let R be a commut. ring with a unity $1 \in R$.

$H_n(M|x;R) \cong \mathcal{R} \longleftarrow$ free R-module of rank 1.

A local R-orientation at x is a choice of a generator $u \in \mathcal{R}$, i.e. $\mathcal{R} = R \cdot u$.

Of course two generators $u, v \in \mathcal{R}$ differ by an invertible element

$v = \sigma \cdot u$, $\sigma \in R$ invertible.

Define $\tilde{M}_R$ similarly to $\tilde{M}_{\mathbb{Z}}$.

<u>Def.</u> An R-orientation on M, is a section $\mu : M \longrightarrow \tilde{M}_R$ s.t. $\forall x \in M$

μ_x is a generator of $H_n(M|x;R)$.

(Exc. This def. is equiv. to the prev. one for $R = \mathbb{Z}$.)

Rem. $H_n(M|x;R) \cong H_n(M|x)\otimes R$. $\Rightarrow$ inside $\tilde{M}_R$ we have $\tilde{M}_r \subset \tilde{M}_R$ $\forall\, r\in R$

$$\tilde{M}_r = \{(x, \pm\mu_x \otimes r) : x\in M\}.$$

μ_x ← gener. of $H_n(M|x)$

Note that if $2r=0$ (i.e. $r=-r$) $\Rightarrow$ $\tilde{M}_r = M$.

If $2r\neq 0$ $\Rightarrow$ $\tilde{M}_r \approx \tilde{M}$.

Conclusion. 1) If M is orientable then it is R-orientable $\forall$ ring R.

2) Let M be ~~an~~ a non-orientable manif. and R a ring with a unit of order 2 (i.e. $2=0$ in R) $\Rightarrow$ M is R-orientable. In particular any manifold is $\mathbb{Z}_2$-orientable.

Conclusion. 1) If M is orientable then it is R-orientable $\forall$ ring R.

2) Let M be a non-orientable manif. and R a ring with a unit of order 2 (i.e. $2=0$ in R) $\Rightarrow$ M is R-orientable. In particular any manifold is $\mathbb{Z}_2$-orientable.

Thm. Let M be a compact connected n-manifold.

1) If M is R-orientable then the map

$$H_n(M;R) \xrightarrow{i_x} H_n(M|x;R) \cong R \quad \text{is an iso. } \forall x \in M.$$

2) If M is not R-orientable, then $\forall x \in M$ the map

$$H_n(M;R) \xrightarrow{i_x} H_n(M|x;R) \cong R \quad \text{is injective}$$

and its image is $\{a \in R : 2a = 0\}$.

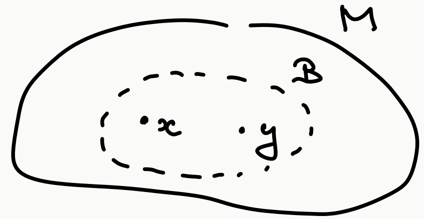
3) $H_i(M;R) = 0 \quad \forall\ i > n$.

So, if M is orientable $\Rightarrow H_n(M;\mathbb{Z}) \cong \mathbb{Z}$
if not $\Rightarrow H_n(M;\mathbb{Z}) = 0$.

Remarks. 1) Suppose M is R-orientable and let μ be an R-orientation.

Let $x \in M$ and consider $\mu_x \in H_n(M|x;R)$. By (1) of the Thm. we get a class $a^x \in H_n(M;R)$ s.t. $a^x \xmapsto{L_x} \mu_x$.

Consider $y \in M$, lying in the same ball chart as x.



$$\begin{tikzcd}
& H_n(M;R) \arrow[dl, "L_x"'] \arrow[d] \arrow[dr, "L_y"] & \\
H_n(M|x;R) & H_n(M|B;R) \arrow[l] \arrow[r] & H_n(M|y;R)
\end{tikzcd}$$

Consider $\mu_y \in H_n(M|y;R)$ coming from μ. $\Rightarrow$ $L_y(a^x) = \mu_y$.

If M is connected, all the above works even if x & y are not in the same ball chart!

Also, if $a \in H_n(M;R)$ a generator $\Rightarrow$ $M \ni x \longmapsto \mu_x := L_x(a)$ is an orientation.

So R orientations $\longleftrightarrow$ generators of $H_n(M;R)$.
on a compact M

A choice of a generator, (in case M is compact + orientable) ~~of~~ of $H_n(M;R)$ is called a fundamental class.

Notation: ~~Alle~~ Let M be a compact R-oriented n-manifold.

We denote by $[M] \in H_n(M;R)$ the fundamental class corresponding to the given orientation.

2) If M, an n-manifold, has a class $a \in H_n(M;R)$ s.t. ~~a~~ a induces an orientation by $x \longmapsto L_x(a)$, then M is compact.

Proof. Let σ be an n-cycle representing a.

Clearly image(σ) = compact. So if $x \in M \setminus \text{image}(\sigma)$

(union of the images of the simplices participating in σ)

$\Rightarrow L_x([\sigma]) = 0 \in H_n(M|x;R)$.

$\Rightarrow \text{image}(\sigma) = M$.

$\blacksquare$

To prove the Thm. we need the following lemma:

<u>Lemma.</u> Let M be an n-manifold. Let $A \subset M$ be a compact subset. Then:

1) If $M \ni x \longmapsto \alpha_x \in H_n(M|x;R)$ is a section of $\tilde{M}_R \longrightarrow M$

then $\exists$ an unique $\alpha_A \in H_n(M|A;R)$ s.t. $L_x(\alpha_A) = \alpha_x \quad \forall\, x \in A$.

2) $H_i(M|A;R) = 0 \quad \forall\, i > n$.

<u>Proof of the Thm. (assuming the Lemma).</u> By assumption $M =$ compact, so we can take $A = M$ in the Lemma. $H_k(M|A;R) = H_k(M,\phi;R) = H_k(M;R)$.

$\Rightarrow$ (3) of the Thm. follows from the Lemma.

Denote by Γ_R the set of sections of $\tilde{M}_R \longrightarrow M$. Note that Γ_R is an R-module (we can add sections and also multiply a sect. by $r \in R$).

(exc. these operations preserve contin.)

We have a homo. $H_n(M;R) \xrightarrow{\Theta} \Gamma_R$

$$H_n(M;R) \ni a \xmapsto{\ \Theta\ } \left(M \ni x \mapsto L_x(a)\right) \in \Gamma_R .$$ By the Lemma, Θ is an iso.

Pick $x_0 \in M$. We have a "restriction" map

$$\begin{array}{ccccc} \rho : \Gamma_R & \longrightarrow & H_n(M|x_0;R) = \left(\tilde{M}_R\right)_{x_0} \\ \Psi & & \cong \\ s & \longmapsto & s_{x_0} \qquad\qquad R \end{array}$$

$\leftarrow$ the fiber of $\tilde{M}_R$ over x_0.

If M is R-orientable then ρ is an iso. (b.c. $\tilde{M}_R \to M$ is a covering).
(& M is path-connect.)

$$\Rightarrow \quad \begin{tikzcd} H_n(M;R) \arrow[r, "\Theta", "\cong"'] \arrow[rr, bend right, "L_x"'] & \Gamma_R \arrow[r, "\rho", "\cong"'] & H_n(M|x;R) \cong R \end{tikzcd}$$ is an iso. $\forall x \in M$.

If M is <u>not</u> R-orientable, then $\Gamma_R \xrightarrow{\rho} H_n(M|x;R)$ is only injective

(b.c. of uniq. of lifts in covering spaces)

clearly, $\text{image}(\rho) = \{a \in H_n(M|x;R) : -a = a\}$,
b.c. $\forall r \in R$ with $2r \neq 0$ we have $\tilde{M}_r \approx \tilde{M}$.

$\Big\}$ exc.

$\blacksquare$

To prove the Lemma we'll need the following version of M-V LES:

<u>Thm.</u> Let X be a space, $Y \subset X$ a subspace.

Let
$$\begin{matrix} Q, & R & \subset X \\ \cup & \cup & \\ S, & T & \subset Y \end{matrix} \qquad \text{s.t.} \qquad \begin{matrix} \mathrm{Int}(Q) \cup \mathrm{Int}(R) = X \\ \mathrm{Int}(S) \cup \mathrm{Int}(T) = Y. \end{matrix}$$

Then $\exists$ a LES

$$\cdots \longrightarrow H_k(Q \cap R, S \cap T) \xrightarrow{\Phi} H_k(Q, S) \oplus H_k(R, T) \xrightarrow{\Psi} H_k(X, Y) \longrightarrow \cdots$$

where $\Phi(x) = (x, -x)$, $\Psi(x, y) = x + y$. The Thm. works with coeffs. in any ab. group. <u>Proof.</u> Exc., see also Hatcher.

<u>Cor.</u> Let M be an n-manif., $A, B \subset M$ compact. Then we have a LES

$$\cdots \longrightarrow H_k(M \mid A \cup B) \xrightarrow{\Phi} H_k(M \mid A) \oplus H_k(M \mid B) \xrightarrow{\Psi} H_k(M \mid A \cap B) \longrightarrow \cdots$$

<u>Proof.</u> Take $Q = R = M = X$, $Y = M \setminus (A \cap B)$, $S = M \setminus A$, $T = M \setminus B$. $\blacksquare$

<u>Notation.</u> $B \subset A \subset X$. $H_k(X|A) \xrightarrow{L_{A,B}} H_k(X|B)$

the map induced from the inclusion $(X, X\setminus A) \longrightarrow (X, X\setminus B)$.

<u>Proof of Lemma.</u> We omit R from the notation.

<u>Step 1.</u> If the Lemma holds for $A, B \subset M$ and also for $A \cap B$

$\Rightarrow$ the Lemma holds also for $A \cup B$.

<u>Proof.</u> We'll use the M-V we've seen earlier this lecture.

$$\cdots \longrightarrow H_k(M|A\cup B) \xrightarrow{\Phi} H_k(M|A) \oplus H_k(M|B) \xrightarrow{\Psi} H_k(M|A\cap B) \longrightarrow \cdots$$

As $H_k(M|A\cap B) = 0$ $\forall$ $k \geq n+1$ by assumption, hence we get an ex. seq.:

$$0 \longrightarrow H_n(M|A\cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A\cap B)$$

$\Phi(\alpha) = (\alpha, -\alpha)$ (formally $\Phi(\alpha) = (L_{A\cup B, A}(\alpha), -L_{A\cup B, B}(\alpha))$)

$\Psi(\alpha,\beta) = \alpha + \beta$ (—— " —— $\Psi(\alpha,\beta) = \quad \ldots\ldots$).

We know, by assumption, that $H_k(M|A) = H_k(M|B) = 0 \quad \forall\ k \geq n+1$

$\Rightarrow H_k(M|A\cup B) = 0 \quad \forall\ k \geq n+1$. This proves (2) of the Lemma for $A\cup B$.

If $x \longmapsto \alpha_x$ is a section of $\tilde{M}_R \longrightarrow M$, then by assumption

$\exists\ \alpha_A \in H_n(M|A),\ \alpha_B \in H_n(M|B)$ s.t. $L_{A,x}(\alpha_A) = \alpha_x$, $L_{B,x}(\alpha_B) = \alpha_x$.
$\qquad \forall\ x \in A \qquad\qquad \forall\ x \in B$

Consider $\alpha'_{A\cap B} := L_{A,A\cap B}(\alpha_A)$, $\alpha''_{A\cap B} := L_{B,A\cap B}(\alpha_B)$.

clearly $L_{A\cap B,x}(\alpha'_{A\cap B}) = \alpha_x$, $L_{A\cap B,x}(\alpha''_{A\cap B}) = \alpha_x \quad \forall\ x \in A\cap B$.

By the uniqueness assumption we have $\underbrace{L_{A,A\cap B}(\alpha_A)}_{\alpha'_{A\cap B}} = \underbrace{L_{B,A\cap B}(\alpha_B)}_{\alpha''_{A\cap B}}$.

Denote $\alpha_{A\cap B} := \alpha'_{A\cap B} = \alpha''_{A\cap B}$.

clearly $\Psi(\alpha_A, -\alpha_B) = 0$. By exactness of the M-V seq.,

$\exists\ \alpha_{A\cup B} \in H_n(M|A\cup B)$ s.t. $\Phi(\alpha_{A\cup B}) = (\alpha_A, -\alpha_B)$. $\Rightarrow L_{A\cup B,x}(\alpha_{A\cup B}) = \alpha_x$
$\forall\ x \in A\cup B$.

Uniqueness of $\alpha_{A\cup B}$. Enough to prove that if $L_{A\cup B,x}(\alpha)=0 \ \forall x\in A\cup B$, then $\alpha=0$. Indeed, if $L_{A\cup B,x}(\alpha)=0 \ \forall x\in A\cup B$, then $\alpha_A := L_{A\cup B,A}(\alpha)$ & $\alpha_B := L_{A\cup B,B}(\alpha)$ also satisfy $L_{A,x}(\alpha_A)=0 \ \forall x\in A$ & $L_{B,x}(\alpha_B)=0 \ \forall x\in B$. By the uniq. assumpt. we have $\alpha_A=0$, $\alpha_B=0$. But $(\alpha_A, -\alpha_B)=\Phi(\alpha)$ & Φ is injective

$\Rightarrow \alpha=0$. This completes the proof of step 1.

Lecture #12B.

Lemma. Let M be an n-manifold. Let $A \subset M$ be a compact subset. Then:

1) If $M \ni x \longmapsto \alpha_x \in H_n(M|x;R)$ is a section of $\tilde{M}_R \longrightarrow M$ then $\exists$ an unique $\alpha_A \in H_n(M|A;R)$ s.t. $L_x(\alpha_A) = \alpha_x \ \forall x \in A$.

2) $H_i(M|A;R) = 0 \quad \forall i > n$.

Proof of Lemma. We omit R from the notation.

Step 1. If the Lemma holds for $A, B \subset M$ and also for $A \cap B$

$\Longrightarrow$ the Lemma holds also for $A \cup B$.

Step 2. We'll reduce to proving the Lemma to the case $M = \mathbb{R}^n$.

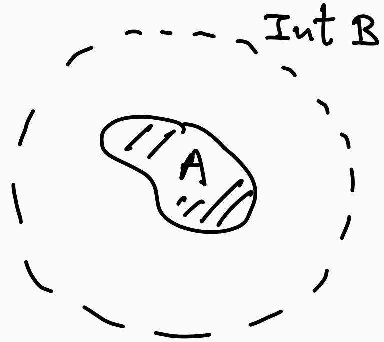
If $A \subset M$ is compact $\Rightarrow$ $A = A_1 \cup \dots \cup A_m$ with A_i = compact $\forall i$, & $A_i \subset$ ball chart $\subset \mathbb{R}^n$. If the result is true for $A_1 \cup \dots \cup A_{m-1}$

& also for A_m & for $(A_1 \cup \dots \cup A_{m-1}) \cap A_m = (A_1 \cap A_m) \cup \dots \cup (A_{m-1} \cap A_m)$

then by step 1, the result holds also for $A_1 \cup \dots \cup A_m$.

So, by induction on m, it is enough to prove the result for $m = 1$, i.e. $A \subset$ ball chart $\subset M$.

Assume that $A \subset \operatorname{Int} B \subset \mathcal{U} \subset M$
$$\begin{array}{ccc} \operatorname{Int} B & \subset & \mathcal{U} \\ \wr\wr & & \wr\wr \\ \operatorname{Int} B^n(\mathbb{R}) & \subset & \mathbb{R}^n \end{array}$$



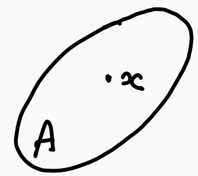
By excision: $H_n(M, M\setminus A) \overset{H_n}{\cong} H_n\big(M\setminus(M\setminus \operatorname{Int} B), M\setminus A\setminus(M\setminus \operatorname{Int} B)\big) =$

$= H_n(\operatorname{Int} B, \operatorname{Int} B\setminus A) \cong H_n(\mathcal{U}|A)$

(the iso's here are induced by inclusions).

so, it's enough to prove the result for $A \subset \mathbb{R}^n$.

<u>Step 3.</u> Assume $M = \mathbb{R}^n$, $A \subset M$ is compact and $A = A_1 \cup \dots \cup A_m$ with A_i = convex $\forall i$. If $m=1$ & A = convex then $H_*(M|A) \xrightarrow{\cong} H_*(M|x)$ for any $x \in A$. (The incl. $(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) \longleftarrow (\mathbb{R}^n, \mathbb{R}^n\setminus A)$ is a homotopy equiv.)



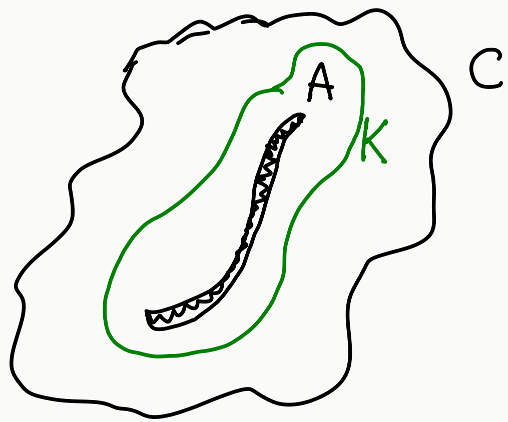
If $A = A_1 \cup \dots \cup A_m$ with A_i = convex, then use induction on m and the previous steps.

Step 4. $M = \mathbb{R}^n$, $A \subset \mathbb{R}^n$ is an arbit. compact subset.

Let $\alpha \in H_i(M|A)$. Let z be a cycle in $S_i(M, M \setminus A)$ with $\alpha = [z]$.

View z also as a chain in $S_i(M)$, and let $C :=$ union of images of all the sing. simplices that participate in ∂z. So $C \subset M \setminus A$.

Clearly C is compact. As C is compact & A too, $\exists\ \delta > 0$ s.t. $\forall\ p \in C, q \in A$, $\operatorname{dist}(p, q) \geq \delta$.



Cover A by finitely many closed balls centered at points of A and with radius $< \frac{\delta}{2}$.

Denote the union of these balls by K.

Note that $C \subset M \setminus K \Rightarrow z$ is also a cycle in $S_i(M, M \setminus K)$.

Put $\alpha_K := [z] \in H_i(M|K)$.

If $i > n$, then by step 3, $\alpha_K = 0$. $\Rightarrow$ $\alpha = L_{K,A}(\alpha_K) = 0$.

$\Rightarrow H_i(M|A) = 0 \quad \forall\ i > n$.

Let $x \longmapsto \alpha_x$ be a sect. of $\tilde{M}_R \longrightarrow M$ $(M = \mathbb{R}^n)$.

Assume that $\alpha_x = L_{A,x}(\alpha)$ $\forall$ $x \in A$ for some $\alpha \in H_n(\mathbb{R}^n | A)$.

We'll show α is unique. Enough to show in case $\alpha_x = 0$ $\forall$ $x \in A$.

<u>claim.</u> assuming $\alpha_x = 0$ $\forall$ $x \in A$ implies $\alpha_x = 0$ $\forall$ $x \in K$.

<u>proof.</u> If $B \subset K$ is one of the balls forming K, then

$$H_n(\mathbb{R}^n | B) \xrightarrow[\cong]{L_{B,x}} H_n(\mathbb{R}^n | x) \quad \text{is an iso } \forall \; x \in B.$$

$\Rightarrow$ $\alpha_x = 0$ $\forall$ $x \in B$.

$\Rightarrow$ $\alpha_x = 0$ $\forall$ $x \in K$. This proves the claim.

Define $\alpha_K \in H_n(\mathbb{R}^n | K)$ as before. By step 3 we get $\alpha_K = 0$.

$\Rightarrow$ $\alpha = L_{K,A}(\alpha_K) = 0$ too. This proves the uniqueness of (1) of the Lemma.

<u>Existence.</u> Pick a huge ball $B(r)$ with $\operatorname{Int} B(r) \supset A$. By step 3, $\exists$ a class $\alpha_{B(r)} \in H_n(\mathbb{R}^n | B(r))$ with $L_{B(r),x}(\alpha_{B(r)}) = \alpha_x$ $\forall$ $x \in B(r)$.

Put $\alpha_A := L_{B(r),A}(\alpha_{B(r)})$.

$\blacksquare$

closed manifold = compact manifold (without boundary).

Cor. Let M be a closed manifold, of dim n, and assume M is connected.

If M is orientable $\Rightarrow$ torsion$\left(H_{n-1}(M)\right) = 0$.

If M is non-orientable $\Rightarrow$ torsion $H_{n-1}(M) = \mathbb{Z}_2$.

Proof. Well known fact: for a closed manif. M, $H_i(M)$ is f. generated $\forall i$.

Recall UCT: $\quad 0 \longrightarrow H_n(M)\otimes R \longrightarrow H_n(M;R) \longrightarrow \mathrm{Tor}\left(H_{n-1}(M), R\right) \longrightarrow 0.$

torsion $H_{n-1}(M) = \bigoplus_{i=1}^{r} \mathbb{Z}_{l_i}$, $\quad l_i \geq 2$, $r \geq 0$. ($r=0$ means torsion $=0$).

Assume M=orientable. If torsion$\left(H_{n-1}(M)\right) \neq 0$, i.e. $r \geq 1$, choose p=prime s.t. $p \mid l_1$. Take $R = \mathbb{Z}_p$.

$$0 \longrightarrow \underset{\mathbb{Z}}{\underset{\|}{H_n(M)\otimes \mathbb{Z}_p}} \longrightarrow \underset{\mathbb{Z}_p}{\underset{\|}{H_n(M;\mathbb{Z}_p)}} \longrightarrow \bigoplus_{i=1}^{r} \underbrace{\mathrm{Tor}\left(\mathbb{Z}_{l_i}, \mathbb{Z}_p\right)}_{\mathbb{Z}_{\gcd(l_i,p)}} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \oplus \cdots \longrightarrow 0 .$$ Impossible.

Assume M is not orientable. Take $R = \mathbb{Z}_m$.

Claim.

$$H_n(M;\mathbb{Z}_m) \cong \{a \in \mathbb{Z}_m : 2a = 0\} = \begin{cases} 0 & m = \text{odd} \\ \underbrace{\{0, \tfrac{m}{2}\}}_{\cong \mathbb{Z}_2} & m = \text{even} \end{cases} \subset \mathbb{Z}_m.$$

proof. $m = \text{odd} \Rightarrow M$ is not $\mathbb{Z}_m$-orientable, b.c. $\forall\, 0 \neq r \in \mathbb{Z}_m$, $\tilde{M}_r \approx \tilde{M}$.

$m = \text{even} > 2 \Rightarrow M$ is not $\mathbb{Z}_m$ orientable.

We get ~~...~~ from UCT: $0 \to \underbrace{H_n(M) \otimes \mathbb{Z}_m}_{0} \to H_n(M;\mathbb{Z}_m) \to \bigoplus_{i=1}^{r} \mathbb{Z}_{\gcd(l_i,m)} \to 0.$

Take $m = \text{odd} \Rightarrow \overbrace{\gcd(l_i, m) = 1}^{\forall i}$ ✓. This holds $\forall\, m = \text{odd} \Rightarrow l_i = 2^{s_i}\ \forall i.$

For $m = \text{even}$, $H_n(M;\mathbb{Z}_m) \cong \mathbb{Z}_2$, hence $\bigoplus_{i=1}^{r} \mathbb{Z}_{\gcd(l_i,m)} \cong \mathbb{Z}_2$.

$\Rightarrow r = 1$ & $l_1 = 2$. $\Rightarrow$ torsion $H_{n-1}(M) = \mathbb{Z}_2$.

∎

<u>Thm.</u> Let M be a connected <u>non-compact</u> n-manif.

Then $H_i(M;R) = 0 \quad \forall \; i \geq n$.

Poincaré Duality.

Thm. Let M be a closed R-oriented n-manifold with fundamental class $[M] \in H_n(M;R)$ (corresp. to the given orientation). Then the map

$$PD : H^k(M;R) \longrightarrow H_{n-k}(M;R), \qquad \alpha \longmapsto \alpha \cap [M]$$

is an isomorphism of R-modules $\forall$ k.

Cohomology with compact support.

G = group of coeffs. $S^{\bullet}(X;G)$. Define $S_c^{\bullet}(X;G) \subset S^{\bullet}(X;G)$ as follows.

$\varphi \in S^i(X;G)$ is called a cochain with compact support, if $\exists$ compact subset $K_\varphi \subset X$ s.t. $\varphi(\sigma) = 0$ $\forall$ chain σ in $X \setminus K_\varphi$.

$S_c^i(X;G) = \{$compactly supported cochains in $S^i(X;G)\}$.

Note that if $\varphi \in S_c^i(X;G) \Rightarrow \delta\varphi \in S_c^{i+1}(X;G)$ b.c. $\langle \delta\varphi, \sigma \rangle = \pm \langle \varphi, \partial\sigma \rangle$ and if σ is in $X \setminus K_\varphi$ then $\partial\sigma$ is in $X \setminus K_\varphi$. So, $S_c^{\bullet} \subset S^{\bullet}$ is a subcomplex.

We write $H^i_c(X;G) := H^i(S^\bullet_c(X;G))$. We call it cohomology with compact support.

Interpretation in terms of direct limits.

Let $\{G_\alpha\}_{\alpha\in I}$ a collection of abelian grps G_α indexed by a directed set I (this means I is partially ordered and $\forall\, \alpha,\beta \in I$, $\exists\, \gamma\in I$ s.t. $\gamma \geq \alpha$ and $\gamma\geq\beta$).

Suppose we are also given $\forall\, \alpha\leq\beta$ in I a homo. $f_{\beta\alpha} : G_\alpha \longrightarrow G_\beta$ s.t. $f_{\alpha\alpha} = id$ $\forall\,\alpha$, if $\alpha\leq\beta\leq\gamma$ then $f_{\gamma\alpha} = f_{\gamma\beta}\circ f_{\beta\alpha}$.

We call such a structure a directed system of groups.

Define $\overbrace{\varinjlim_{\alpha\in I} G_\alpha}^{\text{a grp}}$ as follows:

Consider $\coprod_{\alpha\in I} G_\alpha$. Define an equiv. rel.: $a\in G_\alpha$, $b\in G_\beta$ are declared equiv. $a\sim b$ if $\exists\ \gamma\geq\alpha,\beta$ s.t. $f_{\gamma\alpha}(a) = f_{\gamma\beta}(b)$. $\varinjlim_{\alpha\in I} G_\alpha := \coprod_{\alpha\in I} G_\alpha \big/ \sim$.

Lecture # 13 A.

Interpretation in terms of direct limits. Let $\{G_\alpha\}_{\alpha \in I}$ a collection of abelian grps G_α indexed by a directed set I (I is partially ordered and $\forall\, \alpha, \beta \in I$, $\exists\, \gamma \in I$ s.t. $\gamma \geq \alpha$ and $\gamma \geq \beta$).

Suppose we are also given $\forall\, \alpha \leq \beta$ in I a homo. $f_{\beta\alpha} : G_\alpha \longrightarrow G_\beta$ s.t. $f_{\alpha\alpha} = id$ $\forall\, \alpha$, if $\alpha \leq \beta \leq \gamma$ then $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$.

We call such a structure a directed system of groups.

Define a grp $\varinjlim_{\alpha \in I} G_\alpha$ as follows:

consider $\coprod_{\alpha \in I} G_\alpha$. Define an equiv. rel. : $a \in G_\alpha$, $b \in G_\beta$ are declared equiv. $a \sim b$ if $\exists\, \gamma \geq \alpha, \beta$ s.t. $f_{\gamma\alpha}(a) = f_{\gamma\beta}(b)$. $\varinjlim_{\alpha \in I} G_\alpha := \coprod_{\alpha \in I} G_\alpha \Big/ \sim$.

Claim. $\varinjlim_{\alpha \in I} G_\alpha$ is an ab. group. If $a \in G_\alpha$, $b \in G_\beta$, then $[a] + [b] := [a' + b']$, where $a' = f_{\gamma\alpha}(a)$, $b' = f_{\gamma\beta}(b)$ for some $\gamma \geq \alpha, \beta$. (exc. check details).

Alternative def. $\varinjlim_{\alpha \in I} G_\alpha := \bigoplus_{\alpha \in I} G_\alpha \Big/ H$, H is generated by $\{a - f_{\gamma\alpha}(a) : \forall\, \alpha \in I, a \in G_\alpha, \gamma \geq \alpha\}$.

<u>Remark.</u> If $J \subset I$ is a subset with the property that $\forall\ \alpha \in I$, $\exists\ \beta \in J$ with $\beta \geq \alpha$, then $\varinjlim_{\alpha \in J} G_\alpha \xrightarrow{\cong} \varinjlim_{\alpha \in I} G_\alpha$ is an iso.

In particular, if I has a maximal element μ (i.e. $\mu \geq \alpha\ \forall\ \alpha \in I$), then $\varinjlim_{\alpha \in I} G_\alpha \cong G_\mu$.

The inclusions $G_\alpha \longrightarrow \bigoplus_{\gamma \in I} G_\gamma$ induce homo. $i_\alpha : G_\alpha \longrightarrow \varinjlim_{\gamma \in I} G_\gamma$

& $\forall\ \beta \geq \alpha$ we have

$$\begin{tikzcd} G_\alpha \arrow[dr, "i_\alpha"] \arrow[d, "f_{\beta\alpha}"'] & \\ G_\beta \arrow[r, "i_\beta"'] \arrow[ur, phantom, "\copyright"] & \varinjlim_{\gamma \in I} G_\gamma \end{tikzcd}.$$

Note also that $\forall\ g \in \varinjlim_{\gamma \in I} G_\alpha$, $\exists\ \alpha \in I$, $g_\alpha \in G_\alpha$ s.t. $g = i_\alpha(g_\alpha)$.

Prop. Let $\{G_\alpha\}_{\alpha\in I}$, $\{f_{\beta\alpha}\}_{\beta\geq\alpha}$ be a directed system of ab. groups.

Let A be an ab. group, and $h_\alpha : G_\alpha \longrightarrow A$, $\alpha \in I$, homo.'s s.t.

$\forall\ \beta \geq \alpha$, $h_\beta \cdot f_{\beta\alpha} = h_\alpha$. Then $\exists !$ homo. $h : \varinjlim_{\alpha\in I} G_\alpha \longrightarrow A$

s.t. $h \circ i_\alpha = h_\alpha\ \forall\ \alpha \in I$.

Moreover, $\operatorname{image}(h) = \bigcup_{\alpha\in I} \operatorname{image}(h_\alpha)$

and $\ker(h) = \bigcup_{\alpha\in I} i_\alpha(\ker h_\alpha)$.

$$\begin{tikzcd}
G_\alpha \arrow[dr, "i_\alpha"] \arrow[drr, "h_\alpha"] \arrow[dd, "f_{\beta\alpha}"'] & & \\
& \varinjlim_{\gamma\in I} G_\gamma \arrow[r, dashed, "\exists ! h"] & A \\
G_\beta \arrow[ur, "i_\beta"'] \arrow[urr, "h_\beta"'] & &
\end{tikzcd}$$

Cor. $h : \varinjlim_{\alpha\in I} G_\alpha \longrightarrow A$ is an iso. iff the following <u>two</u> things hold:

1) $\forall\ a \in A$, $\exists\ \alpha \in I$, $g_\alpha \in G_\alpha$ s.t. $h_\alpha(g_\alpha) = a$.

2) If $h_\alpha(g_\alpha) = 0$, then $\exists\ \beta \geq \alpha$ s.t. $f_{\beta\alpha}(g_\alpha) = 0$.

<u>Back to topology.</u> Suppose $X = \bigcup_{\alpha \in I} X_\alpha$ and that $\forall\ \beta \geq \alpha$ we have $X_\beta \supset X_\alpha$. Then the groups $\{H_i(X_\alpha ; G)\}_{\alpha \in I}$ together with

$f_{\beta\alpha} := \left(H_i(X_\alpha ; G) \xrightarrow{inc_*} H_i(X_\beta ; G) \right)$ form a directed system.

Moreover the maps $H_i(X_\alpha ; G) \xrightarrow{inc_*} H_i(X ; G)$ induce a homo.

$$\varinjlim_{\alpha \in I} H_i(X_\alpha ; G) \longrightarrow H_i(X ; G)$$

<u>Cor.</u> Suppose $X = \bigcup_{\alpha \in I} X_\alpha$ as above & suppose that $\forall$ compact subset $K \subset X$, $\exists\ \alpha \in I$ s.t. $X_\alpha \supset K$. Then $\varinjlim_{\alpha \in I} H_i(X ; G) \longrightarrow H_i(X ; G)$ is an iso. $\forall\ i$.

Let X be a space. The compact subsets $K \subset X$ form a directed set (w.r.t. inclusion), b.e. if $K_1, K_2 \subset X$ are compact then $K_1 \cup K_2$ is also compact.

$\forall$ compact subset $K \subset X$ we associate $H^i(X|K;G) := H^i(X, X \setminus K; G)$.

If $K \subset L \subset X$ (with K, L compact), we have the homo.

$$H^i(X|K;G) \xrightarrow{R_{L,K}} H^i(X|L;G) \quad \text{induced by } (X, X \setminus L) \xrightarrow{inc} (X, X \setminus K)$$

<u>claim.</u> $H^i_c(X;G) \cong \varinjlim_{K \subset X} H^i(X|K;G)$.

<u>Proof.</u> $\forall$ compact subset $K \subset X$ we have an obvious homo.

$$H^i(X|K;G) \xrightarrow{h_K} H^i_c(X;G) \quad \text{defined as follows:}$$

define a map $\bar{h}_K : S^i(X, X \setminus K; G) \longrightarrow S^i_c(X;G)$ by:

let $\varphi \in S^i(X, X \setminus K; G)$, i.e. $\varphi : S_i(X)/S_i(X \setminus K) \longrightarrow G$, then

$\tilde{\varphi} := \left(S_i(X) \longrightarrow S_i(X)/S_i(X \setminus K) \xrightarrow{\varphi} G \right)$ is a cochain with compact support.

($\tilde{\varphi}(\sigma) = 0 \ \forall$ chains $\sigma \subset X \setminus K$).

Define $\bar{h}_K(\varphi) := \tilde{\varphi} \in S^i_c(X;G)$. Clearly $\bar{h}_K$ is a chain map.

$\Rightarrow$ it induces a map $h_K : H^i(X|K;G) \longrightarrow H^i_c(X;G)$.

Now $h_L \circ R_{L,K} = h_K$ $\forall$ $K \subset L$ $\Rightarrow$ we get $h: \varinjlim_{K\subset X} H^i(X|K;G) \longrightarrow H^i_c(X;G)$.

Denote by $i_K : H^i(X|K;G) \longrightarrow \varinjlim_{K\subset X} H^i(X|K;G)$ the maps that come with the construction of direct lim.

We'll show h is injective. Let $a \in \ker h_K$. Suppose $a = [\varphi]$.

$h_K(a) = [\tilde{\varphi}]$, where $\tilde{\varphi} = \left(S_i(X) \longrightarrow S_i(X)/S_i(X\setminus K) \xrightarrow{\varphi} G\right)$.

Since $h_K(a) = 0$ $\exists$ cochain $\Psi : S_{i-1}(X) \longrightarrow G$ with ~~some~~ support in in some compact $K' \subset X$ s.t. $\tilde{\varphi} = \Psi \circ \partial$. Clearly $i_K(a) = 0$, b.c.

$R_{K\cup K',K}([\varphi]) = 0$. $\Rightarrow$ $i_K(\ker h_K) = 0$. $\Rightarrow$ $\ker(h) = 0$. $\Rightarrow$ h is injective.

We'll show now that h is surj. Recall $\mathrm{image}(h) = \bigcup_{K\subset X} \mathrm{image}(h_K)$.
Let $b \in H^i_c(X;G)$ and $\varphi : S_i(X) \longrightarrow G$, $b = [\varphi]$.
Assume φ is supported in the compact subset $K \subset X$. $\Rightarrow$ $\varphi|_{S_i(X\setminus K)} \equiv 0$.

$\Rightarrow \varphi$ induces $\bar{\varphi}: S_i(X)/S_i(X \setminus K) \longrightarrow G$ and $\bar{h}_K(\bar{\varphi}) = \varphi. \Rightarrow b = h_K([\bar{\varphi}])$. $\blacksquare$

Example. $H_c^*(\mathbb{R}^n; G)$.

$$H_c^i(\mathbb{R}^n; G) \cong \varinjlim_{K \subset \mathbb{R}^n} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K; G) \cong \varinjlim_{B(R)} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B(R); G)$$

But $H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B(R); G) \cong \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$

exact seq.

$$\left(\begin{array}{c} \underbrace{\tilde{H}^{i-1}(\mathbb{R}^n)}_{0} \to \tilde{H}^{i-1}(\mathbb{R}^n \setminus B(R)) \xrightarrow{\cong} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B(R)) \\ \tilde{H}^{i-1}(\mathbb{R}^n \setminus B(R)) \cong \tilde{H}^{i-1}(S^{n-1}), \quad H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B(R)) \to \underbrace{\tilde{H}^i(\mathbb{R}^n)}_{0} \end{array} \right)$$

Exc. $\forall\ R_1 < R_2$ we have that $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(R_1); G) \longrightarrow H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(R_2); G)$ is an iso.

$$\Rightarrow \quad H_c^i(\mathbb{R}^n; G) \cong \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}.$$

(compare: $H^i(\mathbb{R}^n; G) = \begin{cases} 0 & i \neq 0 \\ G & i = 0 \end{cases}$, $\quad H_i(\mathbb{R}^n; G) = \begin{cases} 0 & i \neq 0 \\ G & i = 0 \end{cases}$)

Rem. $\mathbb{R}^n \simeq pt$ but $H_c^*(\mathbb{R}^n; G) \ncong H_c^*(pt; G)$. How come?

Main problem: if $f: X \to Y$ is a map then f does NOT induce a map $S_c^\cdot(Y;G) \longrightarrow S_c^\cdot(X;G)$ (In order for f to induce such a map we need that $f^{-1}(\text{compact}) = \text{compact}$. Such maps are called proper.)

<u>Back to manifolds.</u> Let M be an R-orientable n-manif, not necess. compact. Fix an R-orientation μ on M. Define $PD: H_c^k(M;R) \longrightarrow H_{n-k}(M;R)$:

$\forall$ $K \subset L \subset M$ compact subsets we have the commut. diag.:

$$\begin{tikzcd}
H^k(M|L;R) \times H_n(M|L;R) \arrow[drr, "\frown"] \arrow[d, shift left, "i_{K,L}"] & & \\
H^k(M|K;R) \times H_n(M|K;R) \arrow[u, shift left, "R_{L,K}"] \arrow[u, phantom, shift right=6ex, "(L_{L,K})"] \arrow[rr, "\frown"'] & & H_{n-k}(M;R)
\end{tikzcd}$$

By a prev. lemma $\exists!$ $\mu_K \in H_n(M|K)$, $\mu_L \in H_n(M|L)$ s.t. $L_{K,x}(\mu_K) = \mu_x$ $\forall x \in K$ & $L_{L,x}(\mu_L) = \mu_x$ $\forall x \in L$. By the uniqueness of μ_K & μ_L we have $i_{K,L}(\mu_L) = \mu_K$.

By naturality of cup product we have: $\alpha \cap i_*(\mu_L) = i^*\alpha \cap \mu_L \quad \forall \alpha \in H^k(M|K;R)$.

where $(M, M \setminus L) \xrightarrow{i} (M, M\setminus K)_\varepsilon$ is the inclusion. But $i_* = i_{K,L}$, $i^* = R_{L,K}$

So we get $\alpha \cap \mu_K = R_{L,K}(\alpha) \cap \mu_L$.

$\Rightarrow$ the homo.'s
$$\begin{aligned} H^k(M|K;R) &\longrightarrow H_{n-k}(M;R) \\ \alpha &\longmapsto \alpha \cap \mu_K \end{aligned}$$

induce a map $\underbrace{\varinjlim_{K \subset M} H^k(M|K;R)}_{\cong \, H^k_c(M;R)} \longrightarrow H_{n-k}(M;R)$

We denote this map $PD: H^k_c(M;R) \longrightarrow H_{n-k}(M;R)$.

Thm. Let M be an R-oriented n-manifold.

Then $PD : H^k_c(M;R) \longrightarrow H_{n-k}(M;R)$ is an iso. $\forall\, k \in \mathbb{Z}$.

For the proof, we need:

Lemma. Suppose M is an R-oriented n-manif. and $M = \mathcal{U} \cup V$, $\mathcal{U}, V =$ open.

Then $\exists$ a commut. diag.:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(\mathcal{U}\cap V) \arrow[r] \arrow[d, "PD_{\mathcal{U}\cap V}"'] & H^k_c(\mathcal{U}) \oplus H^k_c(V) \arrow[r] \arrow[d, "PD_{\mathcal{U}} \oplus -PD_V"] & H^k_c(M) \arrow[r] \arrow[d, "PD_M"] & H^{k+1}_c(\mathcal{U}\cap V) \arrow[r] \arrow[d, "PD_{\mathcal{U}\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(\mathcal{U}\cap V) \arrow[r] & H_{n-k}(\mathcal{U}) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(\mathcal{U}\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

The rows are M-V type of LES's. All coeffs. are in R.

Proof. Recall the rel. M-V: $(X,Y) = (A \cup B, C \cup D)$ with $C \subset A$, $D \subset B$

and s.t. $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, $Y = \mathrm{Int}(C) \cup \mathrm{Int}(D)$, then

$$\cdots \longrightarrow H^k(X,Y) \xrightarrow{\psi} H^k(A,C) \oplus H^k(B,D) \xrightarrow{\varphi} H^k(A\cap B, C \cap D) \longrightarrow \cdots$$

$$\psi(\alpha) = \left(\alpha|_{S.(A)}, \alpha|_{S.(B)}\right), \quad \varphi(\beta, \gamma) = \beta|_{S.(A\cap B)} - \gamma|_{S.(A\cap B)}.$$

We'll use this with $A = B = M$, $C = M \setminus K$, $D = M \setminus L$ where $K \subset \mathcal{U}$, $L \subset V$ are compact. We get the 1'st row of the following diag.:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M|K\cap L) \arrow[r] \arrow[d, "\cong"', "\text{exc.}"] \arrow[dr, phantom, "\text{(c)}"] & H^k(M|K) \oplus H^k(M|L) \arrow[r] \arrow[d, "\cong"', "\text{exc.}"] & H^k(M|K\cup L) \arrow[r, "c'"] \arrow[dd, "(-)\cap\mu_{K\cup L}"] \arrow[ddr, phantom, "3"] & H^{k+1}(M|K\cap L) \arrow[r] \arrow[d, "\cong"', "\text{exc.}"] & \cdots \\
& H^k(\mathcal{U}\cap V|K\cap L) \arrow[r] \arrow[d, "(-)\cap\mu_{K\cap L}"] \arrow[dr, phantom, "1"] & H^k(\mathcal{U}|K) \oplus H^k(V|L) \arrow[d, "(-)\cap\mu_K \oplus -(-)\cap\mu_L"] \arrow[dr, phantom, "2"] & & H^{k+1}(\mathcal{U}\cap V|K\cap L) \arrow[d, "(-)\cap\mu_{K\cap L}"] & \\
\cdots \arrow[r] & H_{n-k}(\mathcal{U}\cap V) \arrow[r] & H_{n-k}(\mathcal{U}) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r, "c''"] & H_{n-k-1}(\mathcal{U}\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

The bottom row is homological M-V. The vertical maps come from the orient., i.e. ~~where~~ $\mu_{K\cap L} \in H_n(\mathcal{U}\cap V|K\cap L)$, $\mu_L \in H_n(V|L)$, $\mu_K \in H_n(\mathcal{U}|K)$ are the restrict. of the given orient. to $K\cap L$, K, L etc.
$\mu_{K\cup L} \in H_n(\mathcal{U}\cup V|K\cup L)$

Claim. square 1, 2, 3 are commut., hence the diag. commutes.

Squares 1 & 2 commute on the chain/cochain level.

We'll check that square ③ also commutes.

$$\begin{tikzcd}
H^k(M|K\cup L) \arrow[r, "c'"] \arrow[dd, "(-)\cap\mu_{K\cup L}"'] & H^{k+1}(M|K\cap L) \arrow[d, "\cong"] \\
 & H^{k+1}(U\cap V|K\cap L) \arrow[d, "(-)\cap\mu_{K\cap L}"] \\
H_{n-k}(M) \arrow[r, "c''"] & H_{n-k-1}(U\cap V)
\end{tikzcd} \qquad (*)$$

<u>The map c':</u> Put $C = M\setminus K$, $D = M\setminus L$. Let $S_\cdot^{C,D} \subset S_\cdot(C\cup D)$ be the subcompl. generated by chains in C & chains in D. $\mathcal{D}_\cdot := S_\cdot(M)/S_\cdot^{C,D}$, $\mathcal{D}^* := \hom(\mathcal{D}_\cdot, R)$.

$\mathcal{D}^*$ = cochains in M that vanish on the chains in C and on the chains in D.

Recall $S_\cdot^{C,D} \xrightarrow{\text{inc}} S_\cdot(C\cup D)$ is a q. iso. $\Rightarrow$ $\mathcal{D}^* \longleftarrow S^\cdot(M, C\cup D) = S^\cdot(M, M\setminus(K\cap L))$.

quasi-iso.

$$0 \longrightarrow \mathcal{D}^* \xrightarrow{\ \psi\ } S^\cdot(M,C) \oplus S^\cdot(M,D) \xrightarrow{\ \varphi\ } S^\cdot(M, C\cap D) \longrightarrow 0$$

$$S^\cdot(M, C\cap D) = S^\cdot(M, M\setminus(K\cup L))$$

$$a \stackrel{\psi}{\longmapsto} (a,a), \quad (b,c) \stackrel{\varphi}{\longmapsto} b-c.$$

From this seq. & the fact that $H^*(S^\cdot(M, C\cup D)) \xrightarrow{\cong} H^*(\mathcal{D}^*)$ we get the seq. on the top, from the beginning of the proof.

How to calculate $c'([\alpha])$ for a cocycle $\alpha \in S^\cdot(M, C\cap D)$.

1'step. $\alpha = \alpha_C - \alpha_D$ with $\alpha_C \in S^\cdot(M, C)$, $\alpha_D \in S^\cdot(M, D)$.

Note that $\delta\alpha_C - \delta\alpha_D = \delta\alpha = 0 \Rightarrow \delta\alpha_C = \delta\alpha_D$.

2'nd step. $(\delta\alpha_C, \delta\alpha_D) = \Psi(\gamma)$. $\gamma = \delta\alpha_C = \delta\alpha_D$.

$$c'([\alpha]) = [\delta\alpha_C] \in H^{k+1}(\mathcal{D}^*) \cong H^{k+1}(M|K\cap L).$$

this is not nec. a cobound. in $\mathcal{D}^*$ b.c. α_C might not belong to $\mathcal{D}^*$ (only $\delta\alpha_C$ is in $\mathcal{D}^*$).

We need to calculate $c'([\alpha]) \cap \mu_{K\cap L}$.

Consider the class $\mu_{K\cup L} \in H_n(M|K\cup L)$. The open sets $U\setminus L$, $U\cap V$, $V\setminus K$ (NOT $U\setminus K$, $U\cap V$, $V\setminus L$!!!) cover $M = U\cup V$. (b.c. $(U\setminus L)\cup(U\cap V) = U$, $(V\setminus K)\cup(U\cap V) = V$)

Using std. M-V (barycentric subdiv. etc.) arguments we can represent $\mu_{K\cup L}$ by a chain $x = x_{U\setminus L} + x_{U\cap V} + x_{V\setminus K}$.

$$\begin{matrix} x_{U\setminus L} & & x_{U\cap V} & & x_{V\setminus K} \\ \cap & & \cap & & \cap \\ S_n(U\setminus L) & & S_n(U\cap V) & & S_n(V\setminus K) \end{matrix}$$

Consider now $\mu_{K\cap L} = \llcorner_{K\cup L, K\cap L}(\mu_{K\cup L}) = [x] \in H_n(M, M\setminus(K\cap L))$

But in $S_n(M)/S_n(M\setminus(K\cap L))$ we have $x_{U\setminus L} = 0$, $x_{V\setminus K} = 0$ $\left(\begin{matrix}\text{b.c. } U\setminus L \subset M\setminus(K\cap L) \\ V\setminus L \subset M\setminus(K\cap L)\end{matrix}\right)$

$\Rightarrow$ $\mu_{K\cap L} = [x_{U\cap V}]$. In a similar way $\mu_K \in H_n(M|K)$ can be written as $\mu_K = [x_{U\setminus L} + x_{U\cap V}]$, $\mu_L = [x_{U\cap V} + x_{V\setminus K}]$.

Let $\alpha \in S^k(M, M\setminus(K\cap L))$ be a cocycle. We've seen that $c'[\alpha] = [\delta\alpha_c]$

So we need to calculat $\delta\alpha_c \cap x_{U\cap V}$.

<u>claim.</u> $[\delta\alpha_c \cap x_{U\cap V}] = (-1)^{k+1}[\alpha_c \cap \partial x_{U\cap V}]$

α_c might not be in $S^{\cdot}(U\cap V|K\cap L)$ so the result here might not be a bndry in $S_{\cdot}(U\cap V)$.

<u>Proof.</u> $\partial(\alpha_c \cap x_{U\cap V}) = \delta\alpha_c \cap x_{U\cap V} + (-1)^k \alpha_c \cap \partial x_{U\cap V}$.

Now $\alpha_c \cap x_{U\cap V} \in S_{\cdot}(U\cap V) \Rightarrow [\delta\alpha_c \cap x_{U\cap V}] = (-1)^{k+1}[\alpha_c \cap \partial x_{U\cap V}] \in H_{n-k-1}(U\cap V)$.

This proves the claim.

<u>Summary:</u>

$$\begin{array}{ccc} [\alpha] & \xmapsto{c'} & [\delta\alpha_c] \\ & & \big\downarrow \cap\mu_{K\cap L} \\ & & (-1)^{k+1}[\alpha_c \cap \partial x_{U\cap V}]. \end{array}$$

consider now the other compos.

$$\begin{array}{ccc} [\alpha] & & \\ \big\downarrow & & \\ [\alpha\cap x] & \xmapsto{c''} & ? \end{array}$$

$$\alpha \cap x = \underbrace{(\alpha \cap x_{U \setminus L})}_{\text{chain in } \mathcal{U}} + \underbrace{(\alpha \cap x_{U\cap V} + \alpha \cap x_{V\setminus K})}_{\text{chain in } V}.$$

<u>Exc.</u> $c''([\alpha\cap x]) = [\partial(\alpha \cap x_{U\setminus L})]$.

To finish the proof of the commut. of (3) (~~of~~ or of $(*)$) we need to show

$$(-1)^{k+1}[\alpha_c \cap \partial x_{U\cap V}] = [\partial(\alpha \cap x_{U\setminus L})].$$

Indeed, $\partial(\alpha\cap x_{U\setminus L}) = \underbrace{\delta\alpha}_{=0} \cap x_{U\setminus L} + (-1)^k \alpha \cap \partial x_{U\setminus L} = (-1)^k(\alpha_c - \alpha_D)\cap \partial x_{U\setminus L} =$

$= (-1)^k \alpha_c \cap \partial x_{U\setminus L} - (-1)^k \underbrace{\alpha_D \cap \partial x_{U\setminus L}}_{=0} =$ (b.c. $\alpha_D \in S^k(M,D)$, $D = M\setminus L$, so $\alpha_D|_{S.(U\setminus L)} \equiv 0$)

$= (-1)^k \alpha_c \cap \partial x_{U\setminus L}$. $(**)$

It remains to show: $(-1)^{k+1}[\alpha_c \cap \partial x_{U\cap V}] = (-1)^k [\alpha_c \cap \partial x_{U\setminus L}]_* \in H_{n-k-1}(U\cap V)$.

Note $\mu_K = [x_{U\cap V} + x_{U\setminus L}] \in H_n(M, M\setminus K) \Rightarrow \partial x_{U\cap V} + \partial x_{U\setminus L} \in S_{n-1}(\underbrace{M\setminus K}_{= C})$.

$$\Rightarrow \alpha_c \cap (\partial x_{U\cap V} + \partial x_{U\setminus L}) = 0, \text{ b.c. } \alpha_c\big|_{S_*(C)} \equiv 0.$$

$\Rightarrow \alpha_c \cap \partial x_{U\cap V} = -\alpha_c \cap \partial x_{U\setminus L}$. From (**) we get

$$[\alpha_c \cap \partial x_{U\cap V}] = -[\alpha_c \cap \partial x_{U\setminus L}] = (-1)^{k+1}[\partial(\alpha \cap x_{U\setminus L})].$$

This completes the proof of the commut. of (3).

<u>Digression.</u> Let $\{G'_\alpha\}, \{G_\alpha\}, \{G''_\alpha\}$, $\alpha \in I$ be directed systems of ab. grps. Suppose $\forall \alpha \in I$ we have an ex. seq. $G'_\alpha \xrightarrow{\varphi_\alpha} G_\alpha \xrightarrow{\psi} G''_\alpha$ and that $\forall$ ~~...~~ $\beta \geq \alpha$: the diag is commut.

$$\begin{tikzcd} G'_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "f'_{\beta\alpha}"'] \arrow[dr, phantom, "\circlearrowleft"] & G_\alpha \arrow[r, "\psi_\alpha"] \arrow[d, "f_{\beta\alpha}"] \arrow[dr, phantom, "\circlearrowleft"] & G''_\alpha \arrow[d, "f''_{\beta\alpha}"] \\ G'_\beta \arrow[r, "\varphi_\beta"] & G_\beta \arrow[r, "\psi_\beta"] & G''_\beta \end{tikzcd}$$

Then $\varinjlim_{\alpha\in I} G'_\alpha \longrightarrow \varinjlim_{\alpha\in I} G_\alpha \longrightarrow \varinjlim_{\alpha\in I} G''_\alpha$ is also exact.

We'll apply this to the diag:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(U\cap V|K\cap L) \arrow[r] \arrow[d] & H^k(U|K)\oplus H^k(V|L) \arrow[r] \arrow[d] & H^k(M|K\cup L) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & \cdots
\end{tikzcd}$$

Apply $\varinjlim_{(K,L)}$ where $K\subset U$, $L\subset V$ are compact & $(K',L')\lesssim (K'',L'')$ if $K'\subset K''$, $L'\subset L''$.

<u>claim.</u> $\varinjlim_{(K,L)} H^k(M|K\cup L)\cong \varinjlim_{\substack{A\subset M\\ \text{compact}}} H^k(M|A)$

<u>proof.</u> $\forall$ $A\subset M$ compact, $\exists$ $K\subset U$, $L\subset V$ compact s.t. $A\subset K\cup L$

Just cover $A\cap U$ by open balls $\bigcup_\alpha B'_\alpha$ with $\overline{B}'_\alpha\subset U$
and —"— $A\cap V$ —"— $\bigcup_\beta B''_\beta$ —"— $\overline{B}''_\beta\subset V$

Now take a finite subcovering of $\bigcup_\alpha \overline{B}'_\alpha \cup \bigcup_\beta \overline{B}''_\beta$ that covers A. This proves the claim.

Conclusion: $\varinjlim_{(K,L)} H^k(M|K\cup L)\cong H^k_c(M)$.

Finally, $\varinjlim_{(K,L)} H^k(U \cap V \mid K \cap L) \cong \varinjlim_{\substack{B \subset U \cap V \\ \text{compact}}} H^k(U \cap V \mid B) \cong H^k_c(U \cap V)$.

$\blacksquare$

Thm. Let M be an R-oriented n-manifold.

Then $PD: H^k_c(M;R) \longrightarrow H_{n-k}(M;R)$ is an iso. $\forall\, k \in \mathbb{Z}$.

Lemma. Suppose M is an R-oriented n-manif. and $M = \mathcal{U} \cup V$, $\mathcal{U}, V =$ open.

Then $\exists$ a commut. diag. with rows being M-V type LES's (coeffs. in R):

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(\mathcal{U}\cap V) \arrow[r] \arrow[d, "PD_{\mathcal{U}\cap V}"'] & H^k_c(\mathcal{U}) \oplus H^k_c(V) \arrow[r] \arrow[d, "PD_{\mathcal{U}} \oplus -PD_V"] & H^k_c(M) \arrow[r] \arrow[d, "PD_M"] & H^{k+1}_c(\mathcal{U}\cap V) \arrow[r] \arrow[d, "PD_{\mathcal{U}\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(\mathcal{U}\cap V) \arrow[r] & H_{n-k}(\mathcal{U}) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(\mathcal{U}\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Proof of the Thm.

Prop. 1. If $M = \mathcal{U} \cup V$ & if $PD_{\mathcal{U}}$, PD_V and $PD_{\mathcal{U}\cap V}$ are all iso's

then PD_M is also an iso.

Proof. The lemma above + 5 lemma.

Prop 2. Suppose I is a directed set and $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ are open subsets of M s.t. $\alpha \leq \beta \Rightarrow \mathcal{U}_\alpha \subset \mathcal{U}_\beta$. Assume also that $\bigcup_{\alpha \in I} \mathcal{U}_\alpha = M$.

If $PD_{\mathcal{U}_\alpha}$ is an iso. $\forall\, \alpha$, then PD_M is an iso.

Proof. $H^k_c(\mathcal{U}_\alpha) \cong \varinjlim_{\substack{K\subset \mathcal{U}_\alpha \\ \text{compact}}} H^k(M|K)$.

$$H^k(M|K) \underset{\text{excis.}}{\cong} H^k(\mathcal{U}_\alpha|K)$$

Note that if $\alpha \leq \beta$ we have $H^k_c(\mathcal{U}_\alpha) \longrightarrow H^k_c(\mathcal{U}_\beta)$ (b.c. $S^k_c(\mathcal{U}_\alpha) \hookrightarrow S^k_c(\mathcal{U}_\beta)$, b.c. if $K \subset \mathcal{U}_\alpha$ is compact $\Rightarrow K \subset \mathcal{U}_\beta$ is compact too)

Now $\varinjlim_{\alpha\in I} H^k_c(\mathcal{U}_\alpha) = \varinjlim_{\alpha \in I} \varinjlim_{\substack{K\subset\mathcal{U}_\alpha \\ \text{compact}}} H^k(\mathcal{U}_\alpha|K) \cong$

$$\cong \varinjlim_{\substack{K\subset M \\ \text{compact}}} H^k(M|K) \cong H^k_c(M).$$

$$\Rightarrow \quad \begin{tikzcd} \varinjlim_{\alpha\in I} H^k_c(\mathcal{U}_\alpha) \arrow[r, "\cong"] \arrow[d, "PD_{\mathcal{U}_\alpha}", "\cong"'] & H^k_c(M) \arrow[d, "PD_M"] \\ \varinjlim_{\alpha\in I} H_{n-k}(\mathcal{U}_\alpha) \arrow[r, "\cong"] & H_{n-k}(M) \end{tikzcd} \quad \Longrightarrow \quad PD_M \text{ is also an iso.}$$

every compact subset in M must be contained in some $\mathcal{U}_\alpha$ and we had a lemma saying that in such a case $\varinjlim_{\alpha} H_*(\mathcal{U}_\alpha) \cong H_*(M)$

$\blacksquare$

Step 1. $M = \mathbb{R}^n$. We saw that

$$\begin{tikzcd} H^k_c(\mathbb{R}^n) \arrow[r, phantom, "\cong"] \arrow[d, "PD"] & H^k(B^n, \partial B^n) \arrow[r, phantom, "\cong"] & H^k(\Delta^n, \partial\Delta^n) \arrow[d, "(-)\cap\mu"] \\ H_{n-k}(\mathbb{R}^n) \arrow[rr, phantom, "\widetilde{=}"] & & H_{n-k}(\Delta^n) \end{tikzcd}$$

μ is a generator $\in H_n(\Delta^n, \partial\Delta^n) \Rightarrow \mu = r\cdot[\sigma]$, where $\sigma : \Delta^n \longrightarrow \Delta^n$ is the id and $r \in R$ is invertible.

We need to check that that $(-)\cap\mu$ is an iso.

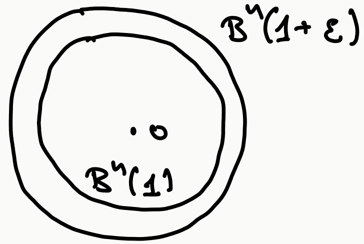
Clearly the only non-trival degree to check is $k=n$.

$H^n(\Delta^n, \partial\Delta^n) = R\cdot[\varphi]$, where φ is an n-cocyle s.t. $\varphi(\sigma) = 1$. (e.g. by UCT),

So $[\varphi]\cap\mu = r[\varphi\cap\sigma] = \pm r[\sigma(e_0)] =$ a generator of $H_0(\Delta^n; R)$.

$H_n(\mathbb{R}^n | B) = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong$

$\cong H_n(B^n(1+\varepsilon) | B^n(1))$



And similarly for cohomology

__Step 2.__ Let $M \subset \mathbb{R}^n$, and assume $M = \bigcup_{i \in I} \mathcal{U}_i$ with I = finite & $\mathcal{U}_i$ = open convex $\forall\, i \in I$. By step 1, $PD_{\mathcal{U}_i}$ is an iso. b.c. $\mathcal{U}_i \approx \mathbb{R}^n$. Now use induction on $|I|$: suppose $I = \{1, \dots, q\}$, put $V_q := \mathcal{U}_1 \cup \dots \cup \mathcal{U}_{q-1}$.

By induction PD is an iso. for V_q & $V_q \cap \mathcal{U}_q$ (and of course $\mathcal{U}_q$ too)

$$V_q \cap \mathcal{U}_q = \Big((\mathcal{U}_1 \cap \mathcal{U}_q) \cup \dots \cup (\mathcal{U}_{q-1} \cap \mathcal{U}_q) \Big)$$

convex

since both V_q & $V_q \cap \mathcal{U}_q$ are unions of at most $q-1$ open convex subsets, $\Longrightarrow$ PD is an iso. also for $V_{q+1} = V_q \cup \mathcal{U}_q$.

__Step 2'.__ $M = \bigcup_{i \in I} \mathcal{U}_i$ with $\mathcal{U}_i$ = open, convex $\subset \mathbb{R}^n$, I = countable.

w.l.o.g. $I = \mathbb{N}$. $\forall\, k \in \mathbb{N}$, put $V_k := \mathcal{U}_1 \cup \dots \cup \mathcal{U}_k$. By step 2, PD is an iso. for V_k, $\forall\, k$. Now $M = \bigcup_{k \in \mathbb{N}} V_k$, so PD_M is an iso. by Prop. 2.

<u>Step 2''.</u> $M \subset \mathbb{R}^n$ is an open subset.

The top. of M has a countable basis consisting of balls.
So by step 2' we are done.

<u>Step 3.</u> $M = \bigcup_{i \in I} \mathcal{U}_i$ with $\mathcal{U}_i$ homeomorphic to open subset in $\mathbb{R}^n$
& I = countable. (we do NOT assume $M \subset \mathbb{R}^n$).

The proof is the same as in steps 2, 2', 2''. First prove for I = finite
by induction on I and then for $I = \mathbb{N}$.

<u>Summary.</u> If M can be covered by countably many charts $\Longrightarrow$ PD_M is an iso.

<u>Step 4.</u> M = a general (non-compact) manifold that can NOT be covered
by a countable union of charts.
Use Zorn's lemma. $\mathcal{T} :=$ collection of all open subsets ~~~~ $\mathcal{U} \subset M$
s.t. $PD_{\mathcal{U}}$ is an iso. Define $\mathcal{U}' \leq \mathcal{U}''$ if $\mathcal{U}' \subset \mathcal{U}''$.

If $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is a chain in $\mathcal{T}$, then $\bigcup_{\alpha\in I} \mathcal{U}_\alpha$ is also in $\mathcal{T}$ (by Prop. 2).

So every chain in $\mathcal{T}$ has an upper bound. By Zorn's lemma $\exists$ a max element V in $\mathcal{T}$. Now if $V \subsetneq M$, take a chart $\mathcal{U}$ around $x_0 \in M \setminus V$.

$PD_{\mathcal{U}}$ is an iso $\Rightarrow$ $\mathcal{U}$ is in $\mathcal{T}$. Also $\mathcal{U} \cap V$ is in $\mathcal{T}$ (b.c. $\mathcal{U}\cap V \subset \mathcal{U}$ is open). $\Rightarrow$ By Prop. 1, $\mathcal{U} \cup V$ is also in $\mathcal{T}$. ↯ Contradiction to maximality of V.

$\blacksquare$

Applications.

Thm. Let M be a closed n-manifold. If n = odd, then $\chi(M) = 0$.

Proof. 1) $\overbrace{\text{Assume } M}$ is orientable. Fix an orient. $H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$.

It is well known that $\forall i$, $H_i(M;\mathbb{Z})$ ~~has finite~~ is finitely generated.

$$\chi = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(M;\mathbb{Z}) \overset{PD}{=} \sum_{i=0}^{n} (-1)^i \operatorname{rank} H^{n-i}(M;\mathbb{Z}) \overset{UCT}{=} \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_{n-i}(M;\mathbb{Z}) =$$

$$\underset{k:=n-i}{=} \sum_{k=0}^{n} (-1)^{n-k} \operatorname{rank} H_k(M) \underset{n=odd}{=} -\sum_{k=0}^{n} (-1)^k \operatorname{rank} H_k(M) = -\chi(M). \Rightarrow \chi(M) = 0.$$

2) Assume M is not orientable.

$$H_i(M;\mathbb{Z}) \cong \mathbb{Z}^r \oplus \mathbb{Z}_{l_1} \oplus \dots \oplus \mathbb{Z}_{l_s} \oplus \mathbb{Z}_{odd} \oplus \dots \oplus \mathbb{Z}_{odd}, \quad 2 \le l_j = \text{even}.$$

$$H_{i-1}(M;\mathbb{Z}) \cong \mathbb{Z}^q \oplus \mathbb{Z}_{t_1} \oplus \dots \oplus \mathbb{Z}_{t_p} \oplus \mathbb{Z}_{odd} \oplus \dots \oplus \mathbb{Z}_{odd}$$

where $2 \le t_j$ = even.

$$\boxed{\begin{aligned} &\operatorname{Ext}(\mathbb{Z},-) = 0 \\ &\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, H) \cong \\ &= H/mH \end{aligned}}$$

By UCT: $H^i(M;\mathbb{Z}_2) \cong \mathbb{Z}_2^{\oplus r} \oplus \mathbb{Z}_2^{\oplus s} \oplus \mathbb{Z}_2^{\oplus p}$.

$$H^{i-1}(M;\mathbb{Z}_2) \cong \mathbb{Z}_2^{\oplus q} \oplus \mathbb{Z}_2^{\oplus p} \oplus \dots$$

For a f. generated ab. grp G, denote $|G|_2 = \#$ of $\mathbb{Z}_{\text{even}}$ summands in G.

So $\dim_{\mathbb{Z}_2} H^i(M;\mathbb{Z}_2) = \operatorname{rank} H_i(M) + |H_i(M)|_2 + |H_{i-1}(M)|_2$.

Note that M is $\mathbb{Z}_2$-orientable, so PD holds with $\mathbb{Z}_2$-coeffs.

In the same way as in step 1 (but now with $\mathbb{Z}_2$-coeffs) we get

$$\sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Z}_2} H^i(M;\mathbb{Z}_2) = 0.$$

So, we get $0 = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Z}_2} H^i(M;\mathbb{Z}_2) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(M) +$

$$+ \sum_{i=0}^{n} (-1)^i \left(|H_i(M)|_2 + |H_{i-1}(M)|_2 \right) = \chi(M) +$$

$$\underbrace{-|H_1(M)|_2}_{i=1} + \underbrace{\left(|H_2(M)|_2 + |H_1(M)|_2\right)}_{i=2} - \left(|H_3(M)|_2 + |H_2(M)|_2\right) + \dots + (-1)^n \left(\underbrace{|H_n(M)|_2}_{=0} + |H_{n-1}(M)|_2\right)$$

b.c. M is not orientable.

$\Rightarrow \chi(M) = 0.$ $\blacksquare$

Cup product pairing.

Let M be a closed R-oriented n-manif. with fund. class $[M] \in H_n(M;R)$.

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R, \quad (\varphi, \psi) \longmapsto \langle \varphi \cup \psi, [M] \rangle.$$

This is an R-bilinear form (or pairing).

Def. A R-bilinear pairing $A \times B \xrightarrow{g} R$ (A, B = R-modules)

is called non-singular if the maps

$$A \longrightarrow \hom_R(B,R) \qquad \text{and} \qquad B \longrightarrow \hom_R(A,R)$$

$$a \longmapsto g(a,-) \qquad\qquad b \longmapsto g(-,b)$$

are both iso's.

Cup product pairing.

Let M be a closed R-oriented n-manif. with fund. class $[M] \in H_n(M;R)$.

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R, \quad (\varphi, \psi) \longmapsto \langle \varphi \cup \psi, [M] \rangle.$$

This is an R-bilinear form (or pairing).

Def. A R-bilinear pairing $A \times B \xrightarrow{g} R$ $\quad$ (A, B = R-modules)
is called non-singular if the maps

$$\begin{aligned} A &\longrightarrow \hom_R(B,R) \quad \text{and} \quad & B &\longrightarrow \hom_R(A,R) \\ a &\longmapsto g(a,-) & b &\longmapsto g(-,b) \end{aligned}$$

are both iso's.

Rem. If $R = \mathbb{Z}$, then the cup prod. pairing satisfies

$$H^k(M;\mathbb{Z})_{\text{torsion}} \times H^{n-k}(M;\mathbb{Z}) \longmapsto 0,$$

$$H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z})_{\text{torsion}} \longmapsto 0.$$

If A is a f. gener. ab. grp. $A_{fr} := A/A_{\text{torsion}}$.

So the cup-prod. pairing gives $H^k(M)_{fr} \times H^{n-k}(M)_{fr} \longrightarrow \mathbb{Z}$.

Prop. Let M be a closed R-oriented n-manif. If $R=$ field then the cup-prod. pairing is non-singular. If $R=\mathbb{Z}$, the pairing

$$H^k(M)_{fr}\times H^{n-k}(M)_{fr}\longrightarrow \mathbb{Z}$$ is non-singular.

Proof.

$$\begin{array}{ccccc} H^{n-k}(M;R) & \xrightarrow{\ h\ } & \hom_R(H_{n-k}(M;R),R) & \xrightarrow{\ PD^*\ } & \hom_R(H^k(M;R),R) \\ \psi & \longmapsto & (a\mapsto\langle\psi,a\rangle) & \longmapsto & (\varphi\mapsto \underbrace{\langle\psi,\varphi\cap[M]\rangle}_{=\ \langle\varphi\cup\psi,[M]\rangle}) \end{array}$$

So, the composition $PD^*\circ h$ is exactly

$\psi\longmapsto g(-,\psi)$ where g is the cup prod. pairing.

If $R=$ field $\Rightarrow$ h is an iso. & PD^* is also an iso. $\Rightarrow$ $PD^*\circ h$ is an iso.

The map $H^k(M)\longrightarrow \hom_R(H^{n-k}(M),R)$, $\varphi\longmapsto g(\varphi,-)$ is also an iso.

b.c. $g(\varphi,-)=(-1)^{k\cdot(n-k)}g(-,\varphi)$. This completes the proof for $R=$ field.

The same works for $R=\mathbb{Z}$ if we mod out torsion, b.c. if H is a f.generated ab. group, then $\hom(H,\mathbb{Z})\cong\hom(H_{fr},\mathbb{Z})$. $\blacksquare$

Cor. Let M be a closed, connected, orientable n-manif.

Let $\alpha \in H^k(M;\mathbb{Z})$ be s.t.:

1) α is not torsion (i.e. $l \cdot \alpha \neq 0 \ \forall\, 0 \neq l \in \mathbb{Z}$).

2) α is not divisible in $H^k(M;\mathbb{Z})$, i.e. if $\alpha = l \cdot \alpha'$ with $\alpha' \in H^k(M;\mathbb{Z})$, and $l \in \mathbb{Z}$, then $l = \pm 1$.

Then $\exists\, \beta \in H^{n-k}(M;\mathbb{Z})$ s.t. $\alpha \cup \beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.

If $\mathbb{Z}$ is replaced by a field R, then we need only to assume R-orientability (+ closed + connected) and instead of conditions 1+2, assume $\alpha \neq 0$

Proof. claim: the inclusion $\alpha \in H^k(M;\mathbb{Z})$ induces an injective map

$$\mathbb{Z} \cdot \alpha \hookrightarrow H^k(M;\mathbb{Z})_{fr}.$$ Moreover $H^k(M;\mathbb{Z})_{fr} / \mathbb{Z} \cdot \alpha \cong$ f. gener. free ab. group.

Exc. Prove the claim. (For the 2'nd statement $\longrightarrow$ enough to show that $H^k(M;\mathbb{Z})_{fr} / \mathbb{Z}\alpha$ is torsion free).

$\Rightarrow$ We can write $H^k(M;\mathbb{Z})_{fr} = \mathbb{Z}\alpha \oplus H$ with H = free ab.

(just by splitting the seq. $0 \to \mathbb{Z}\alpha \to H^k(M;\mathbb{Z})_{fr} \to \underbrace{H^k(M;\mathbb{Z})_{fr}/\mathbb{Z}\alpha}_{\cong H} \to 0$)

Fix an orient. on M, ~~and~~ and let $[M]$ be the fund. class corresp. to it.

Choose $F : H^k(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$ s.t. $F(\alpha) = 1$.

By the non-singularity of the cup-prod. pairing, $\exists\ \beta \in H^{n-k}(M;\mathbb{Z})$

s.t. $F = g(-,\beta)$, namely $F(\gamma) = \langle \gamma \cup \beta, [M] \rangle$.

$1 = F(\alpha) = \langle \alpha \cup \beta, [M] \rangle \Rightarrow \alpha\cup\beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.

$\blacksquare$

Applications to calculations.

1. $M = \mathbb{R}P^n$, $R = \mathbb{Z}_2$ (a field). Assume $n \geq 1$. We saw that $H^i(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$ $\forall\ 0 \leq i \leq n$.

Let $\alpha \in H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ be the generator.

Then $H^i(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2\, \underbrace{\alpha_1 \cup \dots \cup \alpha_1}_{x_i}$.

Proof. Denote by $j : \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n$, $[x_0 : \ldots : x_{n-1}] \overset{j}{\longmapsto} [x_0 : \ldots : x_{n-1} : 0]$

claim. $j_* : H_i(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \longrightarrow H_i(\mathbb{R}P^n; \mathbb{Z}_2)$ & $j^* : H^i(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow H^i(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$

are iso's $\forall$ $0 \le i \le n-1$. (Proof: Use cellular homology).

Induction on n. Write the generator of $H^i(\mathbb{R}P^n; \mathbb{Z}_2)$ as α_i

and the generator of $H^i(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$ by β_i. $0 \le i \le n-1$.

$\Rightarrow j^*(\alpha_i) = \beta_i$ by the claim, $\forall$ $0 \le i \le n-1$. Put $q := (j^*)^{-1}$.

So $\alpha_i = q(\beta_i) \underset{\text{induction}}{=} q(\underbrace{\beta_1 \cup \ldots \cup \beta_1}_{\times i}) = q(\beta_1) \cup \ldots \cup q(\beta_1) = \underbrace{\alpha_1 \cup \ldots \cup \alpha_1}_{\times i}$. $\forall$ $0 \le i \le n-1$

$$\left(\begin{array}{l} q(\beta' \cup \beta'') = q(\beta') \cup q(\beta'') \\ \text{if } |\beta'| + |\beta''| \le n-1 \end{array} \right)$$

$\alpha_n \underset{\text{non-degeneracy of cup prod pairing.}}{=} \alpha_1 \cup \alpha_{n-1} = \alpha_1 \cup \underbrace{\alpha_1 \cup \ldots \cup \alpha_1}_{\times (n-1)}$

$\blacksquare$

2. $M = \mathbb{C}P^n$, $R = \mathbb{Z}$. This is a closed orientable $2n$-dim. manifold.

$$H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i = \text{even} \leq 2n \\ 0 & \text{otherwise} \end{cases}.$$

Put $\alpha_{2j} \in H^{2j}(\mathbb{C}P^n; \mathbb{Z})$ to be a generator, $\forall\ 0 \leq j \leq n$.

Then $\alpha_{2j} = \pm \underbrace{\alpha_2 \cup \dots \cup \alpha_2}_{\times j}$. Proof is similar to the case of $\mathbb{R}P^n$.

Degree.

Let M, N be two n-manifolds, $f : M \longrightarrow N$ a map.
Let $y \in N$, $x \in M$ s.t. $f(x) = y$ and assume $\exists$ charts U_x & U_y around x & y and closed ball charts $B_x \subset U_x$, $B_y \subset U_y$ s.t. f takes B_x homeomorphically onto B_y and ∂B_x to ∂B_y.

Assume M & N are oriented with orientations μ^M & μ^N.

$$\mu^M_x \in H_n(M|x) \cong H_n(B_x, \partial B_x) \cong \mathbb{Z}\cdot\mu^M_x \qquad \left(f\big|_{B_x}\right)_*\left(\mu^M_x\right) = \varepsilon\cdot\mu^N_y$$

$$\mu^N_y \in H_n(N|y) \cong H_n(B_y, \partial B_y) \cong \mathbb{Z}\mu^N_y \qquad \text{with } \varepsilon = \pm 1.$$

$\deg_x(f) := \varepsilon$. local degree.

<u>Global degree.</u> M, N orient<u>ed</u> closed n-manifolds, $f: M \longrightarrow N$.

$\deg(f) := d \in \mathbb{Z}$ s.t.

$f_*([M]) = d\cdot[N]$.

$$\begin{array}{ccc} H_n(M;\mathbb{Z}) & \xrightarrow{f_*} & H_n(N;\mathbb{Z}) \\ \| & & \| \\ \mathbb{Z}[M] & & \mathbb{Z}[N] \end{array}$$

<u>Thm.</u> Suppose $f^{-1}(y) = \{x_1, \dots, x_r\}$, $r \geq 0$, and suppose f maps small nbhds of $x_1, \dots, x_n$ homeomorphically (r=0 means $f^{-1}(y) = \phi$) to a nbhd of y, & bndries to bndry, then

$$\deg(f) = \sum_{i=1}^{r} \deg_{x_i}(f).$$

In partic., if f is not surj. $\leadsto$ then $\deg(f) = 0$.

One can define $\deg(f)$ using cohomology. $\quad f: M \to N$

$f^*: H^n(N) \to H^*(M)$. Let $\mu^N \in H^n(N)$ be the unique class s.t.

$\langle \mu^N, [N] \rangle = 1$ $\quad (\mu^N = PD^{-1}[pnt])$. Similarly we have $\mu^M \in H^n(M)$.

Then: $f^*(\mu^N) = d \cdot \mu^M$, where $d = \deg(f)$. $\quad$ (exc.)

<u>Application.</u> Let $f: \mathbb{C}P^n \to \mathbb{C}P^n$. Then $\deg(f) = k^n$ for some $k \in \mathbb{Z}$.

In particular, if f is smooth & y is a reg. value, then

$$\sum_{x \in f^{-1}(y)} \deg_x(f) = k^n. \qquad \Big(\text{If } f \text{ is hol, } \#f^{-1}(y) = k^n \text{ for some } \underset{\mathbb{Z}}{\overset{\cap}{k}} \geq 0\Big).$$

<u>Proof.</u> Let $a \in H^2(\mathbb{C}P^n)$ be a generator s.t. $\underbrace{a \cup \ldots \cup a}_{\times n} = \varepsilon \cdot \mu^{\mathbb{C}P^n}$, $\quad \varepsilon = \pm 1$.

(This is possible b.c. $\underbrace{a \cup \ldots \cup a}_{\times i}$ is a gener. of $H^{2i}(\mathbb{C}P^n)$.)

$f^*(\mu^{\mathbb{C}P^n}) = \varepsilon f^*(\underbrace{a \cup \ldots \cup a}_{\times n}) = \varepsilon \cdot f^*(a) \cup \ldots \cup f^*(a)$. But $f^*(a) = ka$ for some

$k \in \mathbb{Z}$. $\Rightarrow f^*(\mu^{\mathbb{C}P^n}) = \varepsilon \cdot \underbrace{(ka) \cup \ldots \cup (ka)}_{\times n} = \varepsilon \cdot k^n \underbrace{a \cup \ldots \cup a}_{\times n} = k^n \mu^{\mathbb{C}P^n}$. $\blacksquare$