* ex. class ~ every 2 weeks
* new ex. sheet on Wednesdays, hand in on Tue 2 weeks later & new sheet Wednesday a day later
* SAMUpTool (link on metaphor page)

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hints for Exercise sheet 1:

* 2 b) easier to consider a couple of cases: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K)$ & $c=0$, how do we use 2 a) to write it as $\begin{pmatrix} * & \\ & * \end{pmatrix} \cdot \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$?
  what if $c \neq 0$, but $d=0$? etc.

* 2d) $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in SL_2(\mathbb{F}_p)$. Can we find matrices $A, B$ like in part c) s.t.
  $$A^a B^b \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} \bar{1} & * \\ * & * \end{pmatrix}$$
  can we make this $=0$?

* 2e) consider the map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_p)$ from part d)

* 5b) if $\alpha(g,z) = \frac{\gamma(g \cdot z)}{\gamma(z)}$, for $\gamma : X \to \mathbb{C}^\times$, then for $g \in \mathrm{Stab}_{SL_2(\mathbb{R})}(z)$
  $$\alpha(g,z) = 1$$
  $\rightsquigarrow$ can you find some $g \in SL_2(\mathbb{R})$, $z \in \mathcal{H}$ s.t. this does not work?

# § Geometry of the hyperbolic plane

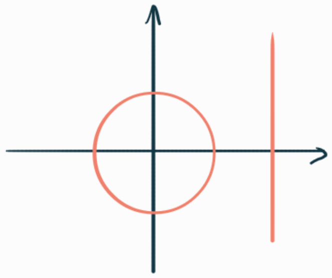
## §1 Isometries

**Def.** Möbius transformation of $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is a biholomorph. rational func.

$$z \mapsto \frac{az+b}{cz+d}$$

w/ $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$ & $\infty \mapsto \frac{a}{c}$, $-\frac{d}{c} \mapsto \infty$

* generalized circle in $\hat{\mathbb{C}}$: either a circle in $\mathbb{C}$ or straight line passing through $\infty$



**Prop.** Möbius transformations preserve generalized circles in $\hat{\mathbb{C}}$

pf. we can decompose

$$z \mapsto \frac{az+b}{cz+d}$$

as composition of inversions $z \mapsto z^{-1}$, scaling $z \mapsto \gamma z$ & translations $z \mapsto z+w$

Say $c \neq 0$.

$$z \mapsto cz+d \mapsto \frac{1}{cz+d} + t = \frac{ctz+dt+1}{cz+d} \mapsto \frac{az + \left(\frac{a}{c}d + \frac{a}{ct}\right)}{cz+d} = \frac{az+b}{cz+d}$$

$t := \frac{-a}{ad-bc}$ multiply by $\frac{a}{ct}$

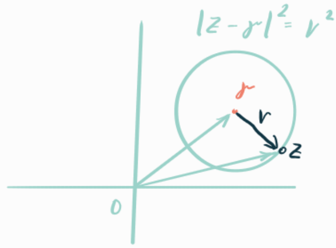
⇝ scaling & translations preserve gener. circles ✓

more precisely, a generalized circle is given by

$$Az\bar{z} + Bz + C\bar{z} + D = 0$$

for $A, D \in \mathbb{R}$, $B$ & $C$ complex conjugates

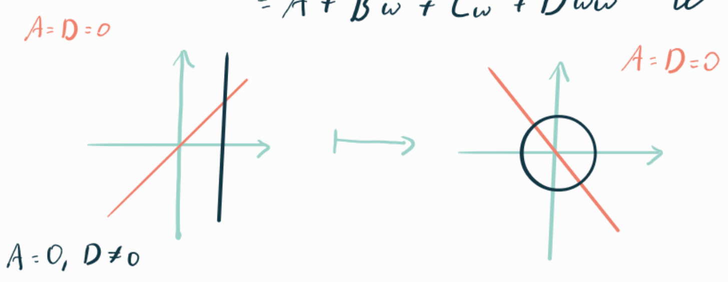
(comes from $r^2 = |z-\gamma|^2 = (z-\gamma)\overline{(z-\gamma)}$)




then $z \mapsto w = \frac{1}{z}$ gives

$$\begin{aligned} 0 &= Az\bar{z} + Bz + C\bar{z} + D \\ &= \frac{A}{w\bar{w}} + \frac{B}{w} + \frac{C}{\bar{w}} + D \\ &= A + B\bar{w} + Cw + Dw\bar{w} \end{aligned}$$

✓ again generalized circle




etc.

□

* Möbius transformations that preserve $\mathbb{R}$ have real coeff.'s, we want def. 1 to preserve vol

## §2 Geodesics („locally shortest path")

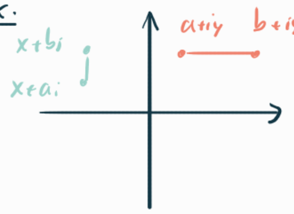
* equip $\mathcal{H}$ w/ Riemann metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

* hyperbolic length of a parametrized piecewise $C^1$ curve $\gamma : [a,b] \to \mathcal{H}$, $t \mapsto x(t) + iy(t)$ is

$$L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

Ex.




$\gamma(t) = t + yi$

$$L(\gamma) = \int_a^b \frac{1}{y}\, dt = b - a$$

$\gamma(t) = x + ti$

$$L(\gamma) = \int_a^b \frac{1}{t}\, dt = \log \frac{b}{a}$$

⇝ the closer to the real axis, the „longer" the segment is

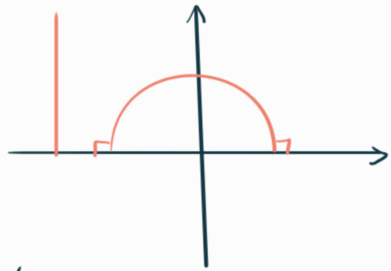
* angle $\theta$ between two vectors $u$ & $v$ in $\mathbb{C}$ is

$$\cos\theta = \frac{u\bar{v}}{|u||v|}$$

(so like Euclidean angle)

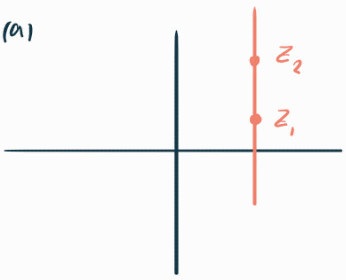
**Def.** $(X, d)$ metric space; the image of a curve $\gamma : [a,b] \to X$ is a geodesic if $\exists \lambda > 0$ s.t. $d(\gamma(t), \gamma(t+\varepsilon)) = \lambda \cdot \varepsilon$ for each small $\varepsilon > 0$.

⇝ geodesics in $\mathcal{H}$: straight lines or semicircles orthogonal to $x$-axis



pf. first, say $z_1 = x + ai$, $z_2 = x + bi \in \mathcal{H}$ w/ $a < b$

(a)

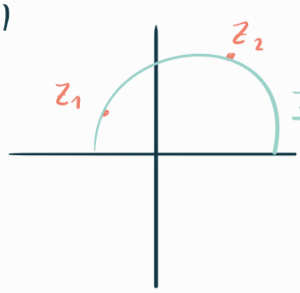



⇒ for any curve $\gamma$ from $z_1$ to $z_2$, we have

$$L(\gamma) = \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt \geq \int \frac{|y'(t)|}{y(t)}\, dt = \int_a^b \frac{dy}{y} = \log\frac{b}{a}$$

↙ hyperbolic length

(b)




∃ semi-circle $L$ w/ center $\alpha$ on $\mathbb{R}$

$\exists g \in PSL_2(\mathbb{R})$ Möbius transf. mapping $L$ to vertical line in $\mathcal{H}$ (choose $g$ s.t. $g(\alpha) = \infty$)

by (a), shortest path $g(z_1)$ to $g(z_2)$ is straight line segment connecting them

$g$ isometry (why?) $\Rightarrow$ preserves length of curves

$\Rightarrow$ path from $z_1$ to $z_2$ along $L$ is distance-minimizing $\square$

$PSL_2(\mathbb{R}) \subset \mathrm{Isom}(\mathcal{H})$

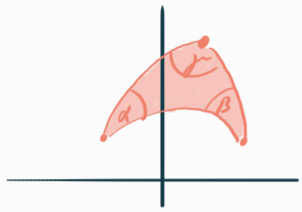
why? $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$. $\rightsquigarrow$ $g'(z) = \frac{1}{(cz+d)^2}$

$$\Rightarrow L(g\circ\gamma) = \int_a^b \frac{|(g\circ\gamma)'(t)|}{\mathrm{Im}(g\circ\gamma(t))}\,dt = \int_a^b \frac{|g'(\gamma(t))|}{\mathrm{Im}(g\circ\gamma(t))}|\gamma'(t)|\,dt =$$

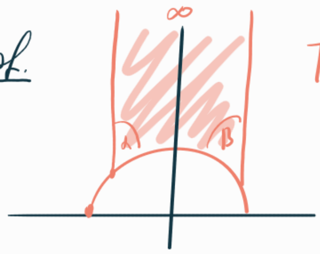
$$= \int_a^b |\gamma'(t)| \cdot \frac{1}{|c\gamma(t)+d|^2}\cdot\left(\frac{1}{|c\gamma(t)+d|^2}\mathrm{Im}\,\gamma(t)\right)^{-1} dt = L(\gamma)$$

**Prop.** (Gauss defect) $T$ hyperbolic triangle w/ angles $\alpha, \beta, \gamma$. $\Rightarrow$

$$\mathrm{area}(T) = \pi - \alpha - \beta - \gamma$$




pf.




$T_0$

by isometries assume assume the other two vertices on geodesic $S^1 \cap \mathcal{H}$

$\rightsquigarrow$ angle at $\infty$ is $0$

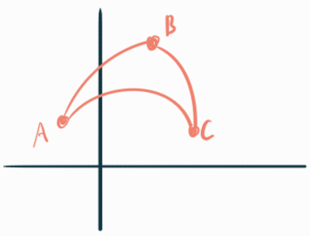
$\rightsquigarrow$ $\mathrm{area}(T_0) \overset{\text{Def}}{:=} \iint_{T_0} \frac{dx\,dy}{y^2}$ ☆

$$= \int_{\cos(\beta)}^{\cos(\pi-\alpha)} \int_{\sqrt{1-x^2}}^{\infty} \frac{dx\,dy}{y^2}$$

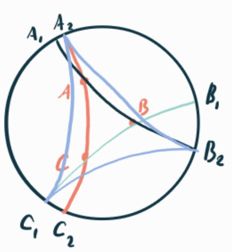
$$= \int_{\sin(\frac{\pi}{2}-\beta)}^{\sin(-\frac{\pi}{2}+\alpha)} \frac{dx}{\sqrt{1-x^2}} \overset{\substack{x=\sin(t)\\ dx=\cos(t)dt}}{=} \frac{\pi}{2} - \beta + \frac{\pi}{2} - \alpha$$

$$= \pi - \alpha - \beta$$

$T$ any triangle w/ angles $\alpha, \beta, \gamma$




$\rightsquigarrow$ Disc




$\rightsquigarrow$ exercise: compare areas $\square$

☆ one has to show that it is invariant under matrices of $SL_2(\mathbb{R})$:

$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A\circ z = A(x_1 + iy_1) = x_2 + iy_2$

$$\rightsquigarrow \int_{A(T)} \frac{dx_1\,dy_1}{y_1^2} \overset{\text{Cauchy Riemann}}{=} \int_T \left|\frac{dA}{dz}\right|^2 \frac{dx_2\,dy_2}{y_1^2} = \int_T \left|\frac{dA}{dz}\right|^2 \left(\frac{|cz+d|^2}{y_2}\right)^2 dx_2\,dy_2 = \int_T \left(\frac{1}{|cz+d|^2}\right)^2 \left(\frac{|cz+d|^4}{y_2^2}\right) dx_2\,dy_2 = \int_T \frac{dx_2\,dy_2}{y_2^2}$$

## §3 Motions in $\mathcal{H}$

set: $N := \left\{ n_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

$A := \left\{ a_y = \begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \mid y > 0 \right\}$

$K := \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$

* seen in lecture: the gp. $PSL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ & $\forall z \in \mathcal{H}$ $\exists g \in PSL_2(\mathbb{R})$ s.t. $g(i) = z$ :

for $z = x + yi$, $g := n_x a_y \Rightarrow g(i) = n_x(iy) = x + iy$ ✓

* exercises: show $\mathrm{Stab}(i) = \{ g \in PSL_2(\mathbb{R}) : g(i) = i \} = K$

**Thm.** The isometries of $PSL_2(\mathbb{R})$ can be classified as follows.

$\pm I \neq g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. $\Rightarrow$ its image in $PSL_2(\mathbb{R})$ is ...

| | if | | conjugate to |
|---|---|---|---|
| ... elliptic | $\lvert a+d \rvert < 2$ | 1 fix. pt. in $\mathcal{H}$ | $k_\theta$ |
| ... parabolic | $\lvert a+d \rvert = 2$ | 1 fix. pt. in $\mathbb{R} \cup \{\infty\}$ | $n_x$ |
| ... hyperbolic | $\lvert a+d \rvert > 2$ | 2 fix. pt. in $\mathbb{R} \cup \{\infty\}$ | $a_y$ |

pf. Consider

$$☆ \quad g(z) = \frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0$$

⇝ solve for $z$

⇝ it has discr. $D = (d-a)^2 + 4bc = (a+d)^2 - 4$

⇝

(a) $\lvert a+d \rvert > 2 \iff D > 0 \iff$ ☆ has 2 sol. in $\mathbb{R} \cup \{\infty\}$

(b) $\lvert a+d \rvert = 2 \iff D = 0 \iff$ –//– unique sol. in $\mathbb{R} \cup \{\infty\}$

(c) $\lvert a+d \rvert < 2 \iff D < 0 \iff$ –//– unique sol. in $\mathcal{H}$

* we are left to determine conjugates

* Case (a): assume up to conjugation the 2 fixed pt.'s are $0$ & $\infty$
  $\leadsto$ motions in $PSL_2(\mathbb{R})$ preserving $0$ & $\infty$ preserve the geodesic between them $\Rightarrow$ get matrices of the form $a_y$

* Case (b): assume up to conjugation, the fixed pt. is $\infty$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \infty \overset{\text{Def.}}{=} \frac{a}{c} = \infty \iff c = 0 \quad (\text{so } a \neq 0)$$

  $\leadsto$ conjugate to $n_x$
  not equal! could be $\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix}$ as well!

* Case (c): conjugate the fixed pt. to $i$, & then note that $\mathrm{Stab}(i) = K$

$\square$