Exercise Sheet 1

1. Let

$$\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$$

be the unit disc in $\mathbf{C}$.

Prove that the maps

$$f\colon \mathbf{H} \to \mathbf{D}, \qquad g\colon \mathbf{D} \to \mathbf{H}$$

defined by

$$f(z) = \frac{z-i}{z+i}, \qquad g(w) = i\frac{w+1}{1-w}$$

are indeed well-defined and are reciprocal conformal equivalences (i.e., they are reciprocal bijective holomorphic maps).

2. a) Let K be a field. Show that the subgroup G of $\mathrm{SL}_2(K)$ generated by the matrices of the form

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad v(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

for $x \in K$ contains the matrices

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

for $a \in K^{\times}$. (Hint: consider a product $u(b)v(c)u(d)v(e)$ and specialize b, c, d and e.)

b) Deduce that $G = \mathrm{SL}_2(K)$. (Hint: use linear algebra to reduce to the previous question.)

c) Let p be a prime number. Show that $\mathrm{SL}_2(\mathbf{F}_p)$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

d) Show that for any prime number p, the map $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{F}_p)$ defined by reduction modulo p is a surjective homomorphism.

e) For any positive integer $q \geq 1$, prove that the subgroups

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid c \equiv 0 \bmod q \right\},$$

$$\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid c \equiv 0 \bmod q,\ a, d \equiv 1 \bmod q \right\},$$

$$\Gamma(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod q \right\},$$

have finite index in $\mathrm{SL}_2(\mathbf{Z})$. Compute this index when q is a prime number. (Hint: compute the size of $\mathrm{SL}_2(\mathbf{F}_p)$.)

3. Let G be a group and let X be a set of which G acts on the left, where the action of $g \in G$ on $x \in X$ is denoted $g \cdot x$. We denote by $\mathcal{C}(X)$ the space of all functions $f\colon X \to \mathbf{C}$.

a) Show that G acts *on the right* on $\mathcal{C}(X)$ by

$$(f \cdot g)(x) = f(g \cdot x)$$

for $f \in \mathcal{C}(X)$, $g \in G$ and $x \in X$.

b) Let $\alpha\colon G \times X \to \mathbf{C}^\times$ be a function. Show that defining

$$(f \bullet g)(x) = \alpha(g, x) f(g \cdot x)$$

(for $f \in \mathcal{C}(X)$, $g \in G$ and $x \in X$) defines an action of G on $\mathcal{C}(X)$ if and only if

$$\alpha(gh, x) = \alpha(h, x)\alpha(g, h \cdot x) \tag{1}$$

for all $(g, h) \in G^2$ and $x \in X$.

c) Let $\gamma\colon X \to \mathbf{C}^\times$ be any function. Show that

$$\alpha(g, x) = \frac{\gamma(g \cdot x)}{\gamma(x)}$$

defines a function with the property of Question b).

4. Let G be a group acting transitively on a non-empty set X. Let $\alpha\colon G \times X \to \mathbf{C}^\times$ be a function satisfying the relation (1) of the previous exercise. We denote again by $\bullet$ the corresponding action of G on $\mathcal{C}(X)$. We also denote by $\mathcal{C}(G)$ the space of functions on G.

Let H be a subgroup of G. We denote by V_H the space of $f \in \mathcal{C}(X)$ such that $f \bullet h = f$ for all $h \in H$.

a) Let $x_0 \in X$ be fixed. Let λ be the linear map

$$\lambda\colon V_H \to \mathcal{C}(G)$$

defined by

$$\lambda(f)(g) = f(g \cdot x_0)\alpha(g, x_0).$$

Show that λ is injective.

b) Show that for all $f \in V_H$, the function $\widetilde{f} = \lambda(f)$ satisfies

$$\widetilde{f}(hg) = \widetilde{f}(g) \tag{2}$$

for all $h \in H$ and $g \in G$.

c) Let K be the stabilizer of x_0 in G. Show that the map $\chi\colon K \to \mathbf{C}^\times$ defined by $\chi(k) = \alpha(k, x_0)$ is a group homomorphism.

d) Show that for all $f \in V_H$, the function $\widetilde{f} = \lambda(f)$ also satisfies

$$\widetilde{f}(gk) = \chi(k)\widetilde{f}(g) \tag{3}$$

for all $k \in K$ and $g \in G$.

e) Show that the image of λ is the set of functions $\widetilde{f}\colon G \to \mathbf{C}$ such that (2) and (3) are valid for all $g \in G$, $h \in H$ and $k \in K$. (Hint: given a function $\widetilde{f}$ which satisfies those conditions, define explicitly a function f so that $\lambda(f) = \widetilde{f}$.)

5. Let $G = \mathrm{SL}_2(\mathbf{R})$, $X = \mathbf{H}$ and consider the action of G on X by Möbius transformations.

a) Show that for any integer $k \in \mathbf{Z}$, we can define

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (cz + d)^k$$

to have an example of the situation of the two previous exercises. In what way is this related to modular forms?

b) Show that if $k \neq 0$, this function α is not of the form given by Question c) of Exercise 3.

Due date: 04.03.2025