D-MATH Number Theory II 05.03.2025
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Exercise Sheet 2

1. a) Show that the measure

$$\mu = \frac{dxdy}{y^2}$$

on $\mathbf{H}$ (with coordinate $z = x + iy$) is invariant under the action of $\mathrm{SL}_2(\mathbf{R})$: for any $g \in \mathrm{SL}_2(\mathbf{R})$ and any μ-integrable function $f \colon \mathbf{H} \to \mathbf{C}$, we have

$$\int_{\mathbf{H}} f(g \cdot z) d\mu(z) = \int_{\mathbf{H}} f(z) d\mu(z).$$

b) Let $f \colon \mathbf{H} \to \mathbf{C}$ be any function which is modular of weight $k \in \mathbf{Z}$. Show that the function ϕ defined on $\mathbf{H}$ by

$$\phi(z) = |f(z)| \mathrm{Im}(z)^{k/2}$$

is modular of weight 0 (i.e., is an $\mathrm{SL}_2(\mathbf{Z})$-invariant function on $\mathbf{H}$).

c) Suppose that f is furthermore meromorphic on $\mathbf{H}$ and modular of weight $k \geq 2$. Show that f is a cusp form if and only if ϕ is bounded on $\mathbf{H}$.

2. The goal of this exercise is to prove that the function Δ defined by

$$\Delta(z) = e(z) \prod_{n \geq 1} (1 - e(nz))^{24}$$

for $z \in \mathbf{H}$ is a cusp form of weight 12, where we recall that $e(z) = e^{2i\pi z}$ for $z \in \mathbf{C}$.

For $z \in \mathbf{C}$ with $\sin(z) \neq 0$, we define

$$\mathrm{cotan}(z) = \frac{\cos(z)}{\sin(z)}.$$

We fix a complex number $\tau \in \mathbf{H}$.

a) Prove that the infinite product converges locally uniformly absolutely, and hence that Δ is a well-defined holomorphic function on $\mathbf{H}$.

b) Show that cotan defines a meromorphic function on $\mathbf{C}$ with simple poles at $z = k\pi$ for $k \in \mathbf{Z}$ with residue 1. Prove that

$$\mathrm{cotan}(z) = -i\Big(1 - \frac{2}{1 - e^{-2iz}}\Big)$$

for $z \in \mathbf{C}$.

c) Let $m \geq 0$ be an integer and define meromorphic functions f_m and g_m by

$$f_m(z) = \mathrm{cotan}((m+\tfrac{1}{2})z)\,\mathrm{cotan}((m+\tfrac{1}{2})z/\tau)$$

and $g_m(z) = z^{-1} f_m(z)$. Show that g_m has

i) simple poles at $\pi k/(m+\frac{1}{2})$ for $k \in \mathbf{Z}$, k non-zero;

ii) simple poles at $\pi k\tau/(m+\frac{1}{2})$ for $k \in \mathbf{Z}$, k integer;

iii) a triple pole at $z = 0$.

d) Show that

$$\mathrm{Res}_{z=\pi k/(m+\frac{1}{2})}\, g_m(z) = \frac{1}{\pi k}\mathrm{cotan}(\pi k/\tau), \qquad \mathrm{Res}_{z=\pi k\tau/(m+\frac{1}{2})}\, g_m(z) = \frac{1}{\pi k}\mathrm{cotan}(\pi k\tau),$$

and

$$\mathrm{Res}_{z=0}\, g_m(z) = -\frac{1}{3}(\tau + \tau^{-1}).$$

e) Let Γ be the polygonal contour in $\mathbf{C}$ joining in counterclockwise order the vertices 1, τ, -1, $-\tau$ and 1 again. Prove that the functions g_m are uniformly bounded on Γ for all m, and prove that

$$\lim_{m\to+\infty} \int_\Gamma g_m(z)dz = \int_1^\tau \frac{dz}{z} - \int_\tau^{-1} \frac{dz}{z} + \int_{-1}^{-\tau} \frac{dz}{z} - \int_{-\tau}^{1} \frac{dz}{z}.$$

(Hint: compute the limit of $g_m(z)$ for z in Γ outside of the vertices.) Deduce the value, as a function of τ, of

$$\lim_{m\to+\infty} \exp\Bigl(3\int_\Gamma g_m(z)dz\Bigr).$$

f) Prove that for all m, we have

$$\int_\Gamma g_m(z)dz = -\frac{2i\pi}{3}(\tau+\tau^{-1}) + 8\sum_{k=1}^{m} \frac{1}{k}\Bigl(\frac{1}{e(-k\tau)-1} - \frac{1}{e(k/\tau)-1}\Bigr).$$

g) Deduce that

$$\lim_{m\to+\infty} \exp\Bigl(3\int_\Gamma g_m(z)dz\Bigr) = \frac{\Delta(-1/\tau)}{\Delta(\tau)},$$

and conclude that $\Delta \in M_{12}^0$. (This proof is due to Siegel.)

Due date: 18.3.2025