Exercise Sheet 3

1. For z a complex number denote by $\mathrm{Im}(z)$ the imaginary part. Define the following functions

$$G_2(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2},$$
$$G_2^*(z) = G_2(z) - \frac{\pi}{2\mathrm{Im}(z)},$$
$$G_{2,\varepsilon}(z) = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^2} \frac{1}{|mz+n|^{2\varepsilon}},$$

for $z \in \mathbf{H}$, where $\varepsilon > 0$ is a parameter.

a) Prove that the series $G_{2,\varepsilon}(z)$ converges absolutely and locally uniformly for $z \in \mathbf{H}$.

b) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}_2(\mathbf{Z})$. Show that

$$G_{2,\varepsilon}(\gamma z) = (cz+d)^2 |cz+d|^{2\varepsilon} G_{2,\varepsilon}(z).$$

for any $z \in \mathbf{H}$.

c) For $\varepsilon > -1/2$ and $z \in \mathbf{H}$, define

$$I(\varepsilon, z) = \int_{\mathbf{R}} \frac{dt}{(z+t)^2 |z+t|^{2\varepsilon}}.$$

Prove that the series

$$\sum_{m=1}^{\infty} I(\varepsilon, mz)$$

converges absolutely and locally uniformly for $\varepsilon > -1/2$ and prove that

$$\lim_{\varepsilon\to 0} \Big(G_{2,\varepsilon}(z) - \sum_{m=1}^{\infty} I(\varepsilon, mz) \Big) = G_2(z).$$

d) Let

$$I(\varepsilon) = \int_{\mathbf{R}} \frac{dt}{(i+t)^2 (1+t)^{2\varepsilon}}$$

for $\varepsilon > -1/2$. Prove that

$$I(\varepsilon, z) = \frac{I(\varepsilon)}{\mathrm{Im}(z)^{1+2\varepsilon}}$$

and prove that the function I is differentiable at 0 with $I'(0) = -\pi$.

e) Deduce that

$$\lim_{\varepsilon\to 0} G_{2,\varepsilon}(z) = G_2^*(z),$$

and that G_2^* is a (non-holomorphic) modular form of weight 2.

f) Conclude that for $z \in \mathbf{H}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we have

$$G_2(gz) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

2. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a finite-index subgroup. We denote by $\overline{\Gamma}$ the image of Γ in $\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$.

A modular form of weight $k \in \mathbf{Z}$ for Γ is a holomorphic function $f\colon \mathbf{H} \to \mathbf{C}$ such that

$$f(gz) = (cz+d)^k f(z)$$

for all $g \in \Gamma$ and $z \in \mathbf{H}$.

a) With the usual rules for $x = \infty$, prove that $\mathrm{SL}_2(\mathbf{Z})$ acts on the set $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

b) Show that this action is transitive for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, and has finitely many orbits in general. Give an example where it is not transitive. (Hint: use a subgroup from Exercise 2 of Exercise Sheet 1.)

By definition, a *cusp* of Γ is an orbit of the action of Γ on $\mathbf{P}^1(\mathbf{Q})$.

c) Let $x \in \mathbf{P}^1(\mathbf{Q})$. Prove that the image in $\overline{\Gamma}$ of the stabilizer Γ_x of x is infinite cyclic, and more precisely that there exists $\sigma_x \in \mathrm{SL}_2(\mathbf{Z})$ and $h \in \mathbf{Z}$ such that $\sigma_x\infty = x$ and

$$\sigma_x^{-1}\Gamma_x\sigma_x = \left\{ \begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\} \quad \text{or} \quad \sigma_x\Gamma_x\sigma_x^{-1} = \left\{ \pm\begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\}.$$

(Hint: consider first the case $x = \infty$.)

d) What are σ_x and h in the case of the unique cusp of $\mathrm{SL}_2(\mathbf{Z})$?

e) Let $f\colon \mathbf{H} \to \mathbf{C}$ be meromorphic and modular of weight k. Show that for every cusp x of Γ, there exists a function $\widetilde{f}_x\colon D^* \to \mathbf{C}$ such that

$$(f \mid_k \sigma_x)(z) = \widetilde{f}_x(e^{2i\pi z/h})$$

for $z \in \mathbf{H}$, where σ_x and h are as in Question c).

One says that f is *holomorphic at the cusp* x if $\widetilde{f}_x$ is holomorphic at 0. The space of all modular forms of weight k for Γ which are holomorphic on $\mathbf{H}$ and at all cusps is denoted $M_k(\Gamma)$.

f) Check that $M_k(\mathrm{SL}_2(\mathbf{Z}))$ coincides with the space M_k of the lecture.

g) Let $C \subset \mathrm{SL}_2(\mathbf{Z})$ be a set of coset representatives of $\Gamma\backslash\mathrm{SL}_2(\mathbf{Z})$. Prove that if $f \in M_k(\Gamma)$, then

$$\prod_{g \in C} f \mid_k g$$

is an element of $M_{|C|k} = M_{|C|k}(\mathrm{SL}_2(\mathbf{Z}))$, which is non-zero if f is non-zero.

h) Deduce that there exists a constant $A_k \geq 0$ such that

$$\sum_{z \in \Gamma\backslash\mathbf{H}} \frac{v_z(f)}{e_z} \leq A_k$$

for all non-zero $f \in M_k(\Gamma)$, where $e_z = |\mathrm{Stab}_z(\Gamma)|$.

i) Prove that $\dim M_k(\Gamma) \leq A_k + 1$.

3. Let $k \geq 2$ be an even integer. Recall that the Bessel function J_k is defined for $z \in \mathbf{C}$ by

$$J_k(z) = \sum_{n \geq 0} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{k+2n}.$$

a) Prove that for all $z \in \mathbf{C}$, we have

$$J_k(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin(\vartheta) - ik\vartheta} d\vartheta,$$

and deduce that $|J_n(x)| \leq 1$ for all $x \in \mathbf{R}$.

b) Prove that the function $f(z) = J_n(z)$ satisfies the differential equation

$$z^2 f''(z) + z f'(z) + (z^2 - n^2) f(z) = 0.$$

4. Let p be a prime number. Recall that the Kloosterman sum $S(m,n;p)$ is defined by

$$S(m,n;p) = \sum_{x \in \mathbf{F}_p^\times} e\Big(\frac{mx + n\bar{x}}{p}\Big), \qquad x\bar{x} \equiv 1 \bmod p.$$

a) Show that $S(m,n;p) \in \mathbf{R}$.

b) Show that

$$\sum_{m,n \in \mathbf{F}_p} |S(m,n;p)|^4 = p^2 N(p)$$

where $N(p)$ is the number of solutions $(a,b,c,d) \in (\mathbf{F}_p^\times)^4$ of the equations

$$\begin{cases} a + b = c + d \\ a^{-1} + b^{-1} = c^{-1} + d^{-1}. \end{cases}$$

c) For $(m_0, n_0) \in (\mathbf{F}_p^\times)^2$, prove that

$$(p-1)|S(m_0, n_0; p)|^4 \leq \sum_{m,n \in \mathbf{F}_p} |S(m, n; p)|^4.$$

d) Deduce that for $(m_0, n_0) \in (\mathbf{F}_p^\times)^2$, we have

$$S(m_0, n_0; p) = O(p^{3/4}).$$

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