Exercise Sheet 4

1. Let $q \geq 1$ be an integer and let $f \in S_k(q)$ be a cusp form of weight k for $\Gamma_0(q)$ (with trivial nebentypus). Let (a_n) be the Fourier coefficients of f at ∞.

Let $\alpha \in \mathbf{R}$.

a) Show that for any $y > 0$ and integer $N \geq 1$, we have

$$\sum_{n \leq N} a_n e(n\alpha) = \int_0^1 f(t + iy + \alpha)\Big(\sum_{1 \leq n \leq N} e(-n(t + iy))\Big)dt.$$

b) Deduce that for $N \geq 1$ and $\alpha \in \mathbf{R}$, we have

$$\sum_{n \leq N} a_n e(n\alpha) = O(N^{k/2} \log N),$$

where the implied constant depends only on f.

2. Let $q \geq 1$ be a prime number and let $f \in S_k(q)$ be a cusp form of weight k for $\Gamma_0(q)$ (with trivial nebentypus). Let

$$f(z) = \sum_{n \geq 1} a_n e(nz)$$

be the Fourier expansion of f at ∞.

Let $r \geq 1$ be a prime number distinct from q and let χ be a non-trivial Dirichlet character modulo r. Define

$$(f \times \chi)(z) = \sum_{n \geq 1} a_n \chi(n) e(nz)$$

for $z \in \mathbf{H}$.

a) Show that the Gauss sum

$$\tau(\chi) = \sum_{x \in \mathbf{Z}/r\mathbf{Z}} \chi(x) e(x/r)$$

satisfies $|\tau(\chi)| = \sqrt{r}$.

b) Show that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{x \in \mathbf{Z}/r\mathbf{Z}} \overline{\chi}(x) e(nx/r)$$

for all integers $n \geq 1$.

c) Show that

$$(f \times \chi)(z) = \frac{1}{\tau(\overline{\chi})} \sum_{0 \leq u \leq r-1} \overline{\chi}(u) f\Big(z + \frac{u}{r}\Big).$$

d) Show that

$$((f \times \chi) \mid_k g)(z) = \chi(d)^2 (f \times \chi)(z)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(qr^2)$.

e) Deduce that $f \times \chi \in S_k(qr^2, \chi^2)$.

3. Let $q \geq 1$ be an integer and let $f \in S_k(q)$ be a cusp form of weight k for $\Gamma_0(q)$ (with trivial nebentypus).

a) For an integer $d \geq 1$, show that

$$g(z) = f(dz)$$

defines a cusp form $g \in S_k(dq)$.

b) Show that if $m \geq 1$ is an integer coprime to dq, we have

$$(T(m)g)(z) = (T(m)f)(dz),$$

where $T(m)$ is the m-th Hecke operator.

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