Exercise Sheet 5

1. Let $k \geq 4$ be an even integer. For any integer $m \geq 0$, we denote by P_m the m-Poincaré series of weight k for $\mathrm{SL}_2(\mathbf{Z})$. We denote by $\langle f_1, f_2 \rangle$ the Petersson inner product for cusp forms of weight k on $\mathrm{SL}_2(\mathbf{Z})$.

a) Let $n \geq 1$ and $m \geq 0$ be integers. Prove that for all $z \in \mathbf{H}$, we have

$$T(n)P_m(z) = n^{k-1} \sum_{g \in N \backslash G_n} (cz+d)^{-k} e(mg \cdot z),$$

where G_n is the set of integral matrices with determinant n and

$$N = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbf{Z} \right\}.$$

b) Let A be a set of matrix representatives of the cosets in $N \backslash \mathrm{SL}_2(\mathbf{Z})$. Prove that the matrices of the form gh, with $g \in \Delta_n$ and $h \in A$ form a set of representatives for $N \backslash G_n$.

c) Deduce that

$$T(n)P_m(z) = n^{k-1} \sum_{\substack{ad=n \\ a,d \geq 1}} \sum_{0 \leq b < d} \sum_{h \in A} (cz+d)^{-k} e(m(ah \cdot z + b)/d),$$

and conclude that

$$T(n)P_m = \sum_{d \mid (n,m)} \left(\frac{n}{d}\right)^{k-1} P_{mn/d^2}.$$

d) Show that the Eisenstein series $E_k = P_0$ is an eigenfunction of all operators $T(n)$ and show that this recovers the Fourier expansion of Eisenstein series.

e) For m, $n \geq 1$, prove that

$$m^{k-1} T(n) P_m = n^{k-1} T(m) P_n.$$

f) Prove that for all integers m, $n \geq 1$ and all $f \in S_k(1)$, we have

$$m^{k-1} \langle T(n)f, P_m \rangle = n^{k-1} \langle T(m)f, P_n \rangle.$$

g) Deduce a new proof of the fact that $T(n)$ is a self-adjoint linear operator for the Petersson inner product.

2. Let $q \geq 1$ be a prime number and χ a non-trivial Dirichlet character modulo q. Let $k \geq 2$ be an integer. Let $f \in S_k(q, \chi)$ and let $a_f(n)$ denote its Fourier coefficients at infinity.

a) Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of $\Gamma_0(q)$ with $c \neq 0$. Prove that for all $z \in \mathbf{H}$, we have

$$(cz)^{-k} f\Big(\frac{a}{c} - \frac{1}{cz}\Big) = \chi(d) f\Big(-\frac{d}{c} + \frac{z}{c}\Big).$$

b) Prove that for all integers $c \geq 1$ divisible by q and integers a, d with $ad \equiv 1 \pmod{c}$, we have

$$(cz)^{-k} \sum_{m \geq 1} a_f(m) e\Big(\frac{am}{c} - \frac{m}{cz}\Big) = \chi(d) \sum_{n \geq 1} a_f(n) e\Big(-\frac{dn}{c} + \frac{nz}{c}\Big)$$

for all z.

c) Deduce that

$$(qz)^{-k} \sum_{m \geq 1} a_f(m) c_q(m) e(-m/(qz)) = \sum_{n \geq 1} a_f(n) \overline{\chi(-n)\tau(\chi)} e(nz/q),$$

where $\tau(\chi)$ is the Gauss sum (see Exercise 2, (a) of Exercise Sheet 4) and

$$c_q(m) = \sum_{\substack{a \bmod q \\ (a,q)=1}} e\Big(\frac{am}{q}\Big).$$

d) Show that if $a_f(n) = 0$ for all n coprime to q, then $f = 0$.

e) Show that if f is an eigenfunction of all Hecke operators $T(n)$ with $(n, q) = 1$, then $a_f(1) \neq 0$. Moreover, prove that if $\widetilde{f} \in S_k(q, \chi)$ is also an eigenfunction of the Hecke operators $T(n)$ with $(n, q) = 1$, with the same eigenvalues as f, then $\widetilde{f}$ is proportional to f. (This is called the *multiplicity one principle.*)

f) Find examples of integers q and k and non-zero cusp forms $f \in S_k(q)$ such that f is an eigenfunction of the Hecke operators $T(n)$ with $(n, q) = 1$, but $a_f(1) = 0$.

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