Exercise Sheet 6

1. Let $k \geq 4$ be an even integer and let $f \in \mathcal{H}_k$ be a Hecke eigenform (i.e., an element of the Hecke basis of $S_k(1)$). Let $\lambda_f(n)$ denote the eigenvalue for the n-th Hecke operator. Let $\delta \geq 0$ be a real number such that $\lambda_f(n) = O(n^\delta)$ for $n \geq 1$.

a) For any prime number p, prove that the power series

$$\sum_{\nu \geq 0} \lambda(p^\nu) X^\nu$$

has radius of convergence $\geq p^{-\delta}$.

b) Deduce that $|\lambda_f(n)| \leq d(n) n^\delta$ for all $n \geq 1$ (here d is the divisor function).

c) Deduce that the following statements are equivalent:

(i) For all cusp forms $g \in S_k(1)$, the Fourier coefficients $a_g(n)$ of g satisfy the estimate $a_g(n) = O(n^{(k-1)/2+\varepsilon})$ for all $\varepsilon > 0$.

(ii) For all $f \in \mathcal{H}_k$ and for all $n \geq 1$, we have $|\lambda_f(n)| \leq d(n) n^{(k-1)/2}$.

2. Let $q \geq 1$ be an integer.

a) Let χ be a Dirichlet character modulo q. For any integer $d \geq 1$, show that the function $\widetilde{\chi}$ defined on integers by

$$\widetilde{\chi}(n) = \begin{cases} \chi(n) & \text{if } (n, qd) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbf{Z}$ is a Dirichlet character modulo qd. It is said to be *induced* by χ. [1]

b) Compute $L(s, \widetilde{\chi})$ in terms of $L(s, \chi)$.

A Dirichlet character modulo q is said to be *primitive* if there does not exist a divisor $d \mid q$ with $d < q$ and a Dirichlet character η modulo d such that χ is induced by η.

c) Explain why the primitive characters modulo a *prime* q are the non-trivial characters modulo q.

d) Show that if χ is any Dirichlet character modulo q, there exists a unique divisor $d \mid q$ (possibly equal to q) and primitive Dirichlet character η modulo d such that χ is induced from η. (Hint: to prove uniqueness, show that if χ is induced from characters η_1 modulo d_1 and η_2 modulo d_2, then it is induced from a character modulo the greatest common divisor $\gcd(d_1, d_2)$.)

[1] Warning: this is not the standard meaning of the word "induce" in representation theory.

e) Let $\varphi^*(q)$ denote the number of primitive characters modulo q. Show that

$$\varphi(q) = \sum_{d|q} \varphi^*(q)$$

and

$$\varphi^*(q) = \sum_{d|q} \mu(d)\varphi(q/d).$$

f) Determine the integers for which $\varphi^*(q) = 0$.

g) Let χ be a primitive Dirichlet character modulo q. Define

$$\tau(\chi) = \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{x}{q}\Big).$$

Show that $\overline{\chi}$ is also primitive, and prove that for any integer a coprime to q, the formula

$$\tau(\overline{\chi})\chi(a) = \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{ax}{q}\Big)$$

holds.

h) Let χ be a primitive Dirichlet character modulo q. Let a be an integer such that $d = (a, q) \geq 2$. Let q' be defined by $q = dq'$.
Prove that for any x' modulo q', we have

$$\sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \bmod q'}} \chi(x) = 0.$$

(Hint: here, you have to use the fact that χ is primitive.)
Deduce that

$$\sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{ax}{q}\Big) = 0.$$

i) Conclude that if χ is a primitive Dirichlet character modulo q, then $\tau(\chi)$ and $\tau(\overline{\chi})$ are non-zero, and the formula

$$\chi(a) = \frac{1}{\tau(\overline{\chi})} \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{ax}{q}\Big)$$

holds for all integers $a \in \mathbf{Z}$. *(This fact is very often the key property of primitive characters.)*

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