Solutions: Exercise Sheet 3

1. For z a complex number denote by $\mathrm{Im}(z)$ the imaginary part. Define the following functions

$$\begin{aligned} G_2(z) &= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}, \\ G_2^*(z) &= G_2(z) - \frac{\pi}{2\mathrm{Im}(z)}, \\ G_{2,\varepsilon}(z) &= \frac{1}{2} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^2} \frac{1}{|mz+n|^{2\varepsilon}}, \end{aligned}$$

for $z \in \mathbf{H}$, where $\varepsilon > 0$ is a parameter.

a) Prove that the series $G_{2,\varepsilon}(z)$ converges absolutely and locally uniformly for $z \in \mathbf{H}$.

Solution:

Let $k > 2$ be an integer and $z \in \mathbf{H}$ arbitrary. Then

$$\sum_{N=1}^{\infty} \sum_{N<|mz+n|\leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n)\in\mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$

Note that

$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$

Thus the above sum is, as $k > 2$

$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$

Now we see that,

$$G_{2,\varepsilon} \leq \sum_{0\leq|mz+n|\leq 1} |mz+n|^{-2-2\varepsilon} + \sum_{1\leq|mz+n|} |mz+n|^{-2-2\varepsilon}.$$

The first sum has a finite number of summands, and the second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of $G_{2,\varepsilon}$ are convergent absolustely and locally uniformly, thus defines a holomorphic function on $\mathbf{H}$.

b) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}_2(\mathbf{Z})$. Show that

$$G_{2,\varepsilon}(\gamma z) = (cz+d)^2 |cz+d|^{2\varepsilon} G_{2,\varepsilon}(z).$$

for any $z \in \mathbf{H}$.

Solution:

To see the transformation law we first note that every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ induces a bijection from $\mathbb{Z}^2 \setminus \{(0,0)\}$ to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma+nc)z + (mb+nd)}{cz+d} = \frac{m'z+n'}{cz+d}.$$

Combining these two facts, we conclude that

$$G_2, \varepsilon(\gamma z) = \sum_{(m',n') \neq (0,0)} \frac{(cz+d)^2 |cz+d|^{2\varepsilon}}{(m'z+n')|m'z+n'|^{2\varepsilon}} = (cz+d)^2 |cz+d|^{2\varepsilon} G_{2,\varepsilon}(z).$$

c) For $\varepsilon > -1/2$ and $z \in \mathbf{H}$, define

$$I(\varepsilon, z) = \int_{\mathbf{R}} \frac{dt}{(z+t)^2 |z+t|^{2\varepsilon}}.$$

Prove that the series

$$\sum_{m=1}^{\infty} I(\varepsilon, mz)$$

converges absolutely and locally uniformly for $\varepsilon > -1/2$ and prove that

$$\lim_{\varepsilon \to 0} \Big(G_{2,\varepsilon}(z) - \sum_{m=1}^{\infty} I(\varepsilon, mz) \Big) = G_2(z).$$

Solution:

Let

$$f(t) := (mz+t)^{-2} |mz+t|^{-2\varepsilon},$$

with implicit dependence on mz. Now as we have proved the absolute convergence of the $\sum f(n)$ we will freely change the order of summations and order of integration and summation, as follows.

$$\begin{aligned}
\tilde{G}_{2,\varepsilon}(z) &= G_{2,\varepsilon}(z) - \sum_{m=0}^{\infty} I(\varepsilon, mz) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} (f(n) - \int_n^{n+1} f(t) dt) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_n^{n+1} (f(n) - f(t)) dt.
\end{aligned}$$

By the mean value theorem on $n \le t \le n+1$ we get that

$$|f(n) - f(t)| \le \sup_{n \le u \le n+1} |f'(u)| \ll |mz+n|^{-3-2\varepsilon}.$$

Hence, the sum is absolutely convergent for $\varepsilon > -1/2$ and thus $\lim_{\varepsilon \to 0} \tilde{G}_{2,\varepsilon}$ exists and defines a holomorphic function. We calculate,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \tilde{G}_{2,\varepsilon}(z) \\
&= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \left[\sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2} + \sum_{n \in \mathbb{Z}} \left(\frac{1}{mz+n+1} - \frac{1}{mz+n} \right) \right] \\
&= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2} \\
&= G_2(z)
\end{aligned}$$

d) Let

$$I(\varepsilon) = \int_{\mathbf{R}} \frac{dt}{(i+t)^2(1+t)^{2\varepsilon}}$$

for $\varepsilon > -1/2$. Prove that

$$I(\varepsilon, z) = \frac{I(\varepsilon)}{\operatorname{Im}(z)^{1+2\varepsilon}}$$

and prove that the function I is differentiable at 0 with $I'(0) = -\pi$.

Solution:

Let $z = x + iy$. Then changing variable $t \mapsto yt - x$ we get that,

$$\begin{aligned}
I(\varepsilon, x+iy) &= \int_{\mathbb{R}} \frac{dt}{(x+t+iy)^2|x+t+iy|^{2\varepsilon}} \\
&= \frac{1}{y^{1+2\varepsilon}} \int_{\mathbb{R}} \frac{dt}{(t+i)^2|t+i|^{2\varepsilon}} = \frac{I(\varepsilon)}{y^{1+2\varepsilon}}.
\end{aligned}$$

Differentiating under the integration sign and then integrating by parts we get that,

$$\begin{aligned}
I'(0) = -\int_{\mathbb{R}} \frac{\log(1+t^2)}{(t+i)^2} dt &= \left. \frac{\log(1+t^2)}{t+i} \right|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2tdt}{(t+i)(1+t^2)} \\
&= -\int_{\mathbb{R}} \frac{1}{(t+i)^2} + \frac{1}{1+t^2} = -\int_{\mathbb{R}} \frac{dt}{t^2+1} = -\pi.
\end{aligned}$$

Using the above two results we compute that,

$$\lim_{\varepsilon \to 0} \sum_{m=1}^{\infty} I(\varepsilon, mz) = \lim_{\varepsilon \to 0} \sum_{m=1}^{\infty} \frac{I(\varepsilon)}{(my)^{1+2\varepsilon}} = \lim_{\varepsilon \to 0} \frac{I(\varepsilon)\zeta(1+2\varepsilon)}{\operatorname{Im}(z)^{1+2\varepsilon}}.$$

Recall the expansion

$$\zeta(1+2\varepsilon) = \frac{1}{2\varepsilon} + O(1).$$

Using that $I(0) = 0$ we have that above limit equals to

$$\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{2\varepsilon \mathrm{Im}(z)^{1+2\varepsilon}} = \frac{I'(0)}{2\mathrm{Im}(z)}.$$

e) Deduce that

$$\lim_{\varepsilon \to 0} G_{2,\varepsilon}(z) = G_2^*(z),$$

and that G_2^* is a (non-holomorphic) modular form of weight 2.

Solution:

Compute that

$$\lim_{\varepsilon \to 0} G_{2,\varepsilon}(z) = \lim_{\varepsilon \to 0} \left(\tilde{G}_{2,\varepsilon}(z) + \sum_{m=1}^{\infty} I(\varepsilon, mz) \right) = G_2(z) - \frac{\pi}{2\mathrm{Im}(z)} = G_2^*(z).$$

Then by part b) together with letting $\varepsilon \to 0$, we obtain that G_2^* transforms like a modular form of weight 2.

f) Conclude that for $z \in \mathbf{H}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we have

$$G_2(gz) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

Solution:

Since $G_2^*(z)$ transforms as a modular form of weight 2, we have that

$$\begin{aligned} G_2(\gamma z) - (cz+d)^2 G_2(z) &= \frac{\pi}{2\mathrm{Im}(\gamma z)} - (cz+d)^2 \frac{\pi}{2\mathrm{Im}(z)} \\ &= \frac{\pi}{2\mathrm{Im}(z)} (|cz+d|^2 - (cz+d)^2) \\ &= \pi i c(cz+d), \end{aligned}$$

concluding the result.

2. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a finite-index subgroup. We denote by $\overline{\Gamma}$ the image of Γ in $\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$.

A modular form of weight $k \in \mathbf{Z}$ for Γ is a holomorphic function $f \colon \mathbf{H} \to \mathbf{C}$ such that

$$f(gz) = (cz+d)^k f(z)$$

for all $g \in \Gamma$ and $z \in \mathbf{H}$.

a) With the usual rules for $x = \infty$, prove that $\mathrm{SL}_2(\mathbf{Z})$ acts on the set $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

Solution:

We can define an action on ∞ by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c},$$

with $ad - bd = 1$, if $c \neq 0$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \infty$ if $c = 0$. and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{-d}{c} = \infty.$$

Since clearly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d} \in \mathbf{Q},$$

this operation is well-defined. What is left to show is if $A, B \in \mathrm{SL}_2(\mathbf{Z})$, then

$$A \cdot (B \cdot z) = (AB) \cdot z.$$

Let

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Applying B to z, we obtain:

$$B \cdot z = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Now applying A to $B \cdot z$:

$$A \cdot (B \cdot z) = A \cdot \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right).$$

By the definition of the Möbius action:

$$A \cdot \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1}.$$

Multiplying numerator and denominator by $c_2 z + d_2$:

$$A \cdot (B \cdot z) = \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)}.$$

Expanding both terms:

$$A \cdot (B \cdot z) = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}.$$

The matrix product AB is:

$$AB = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

Applying AB to z:

$$(AB) \cdot z = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}.$$

Since the expressions for $A \cdot (B \cdot z)$ and $(AB) \cdot z$ are identical, we conclude that:

$$A \cdot (B \cdot z) = (AB) \cdot z.$$

Thus, the Möbius action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{Q}$ is compatible with matrix multiplication.

b) Show that this action is transitive for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, and has finitely many orbits in general. Give an example where it is not transitive. (Hint: use a subgroup from Exercise 2 of Exercise Sheet 1.)

Solution:

We start by showing the action is transitive. By the solution of part a) it is enough to show transitivity for elements in $\mathbf{Q}$. Let $z_1, z_2 \in \mathbf{Q}$ be arbitrary. Choose $a, b \in \mathbf{Z}$ such that $z_1 = \frac{-b}{a}$, and $c, d \in \mathbf{Z}$ such that $ad - bc = 1$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z_1 = 0.$$

Further choose $b', d' \in \mathbf{Z}$ such that $z_1 = \frac{b'}{d'}$, and $a', c' \in \mathbf{Z}$ such that $a'd' - b'c' = 1$. Then

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot 0 = \frac{b'}{d'} = z_2.$$

Hence

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z_1 = z_2,$$

so the action is transitive.

Since the action is transitive, for any $x \in \mathbf{P}^1(\mathbf{Q})$ we have a bijection:

$$\mathrm{SL}_2(\mathbf{Z})/\mathrm{Stab}_x(\mathrm{SL}_2(\mathbf{Z})) \to \mathbf{P}^1(\mathbf{Q})$$

which sends $g\mathrm{Stab}_x(\mathrm{SL}_2(\mathbf{Z}))$ to $g \cdot x$. Since the index of Γ in $\mathrm{SL}_2(\mathbf{Z})$ is finite, we have that the cardinality of $\Gamma \setminus \mathbf{P}^1(\mathbf{Q})$, which is equal to the cardinality of $\Gamma \setminus \mathrm{SL}_2(\mathbf{Z})/\mathrm{Stab}_x(\mathrm{SL}_2(\mathbf{Z}))$, is finite. Hence there are only finitely many orbits.

To give an example of a group for which the action is not transitive, we consider the congruence subgroup $\Gamma_0(p)$ from Exercise sheet 1 (for p a prime). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$, so that $d \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot 0 = \frac{b}{d} \in \mathbf{Q}$$

is a fraction whose denominator is not divisible by p (as otherwise $ad - bc \neq 1$, since $p \mid c$). On the other hand,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a}{c} \in \mathbf{P}^1(\mathbf{Q})$$

is equal to ∞ if $c = 0$ or a fraction in $\mathbf{Q}$ whose denominator is divisible by p. Hence 0 and ∞ lie in different orbits and the action is not transitive.

By definition, a *cusp* of Γ is an orbit of the action of Γ on $\mathbf{P}^1(\mathbf{Q})$.

c) Let $x \in \mathbf{P}^1(\mathbf{Q})$. Prove that the image in $\overline{\Gamma}$ of the stabilizer Γ_x of x is infinite cyclic, and more precisely that there exists $\sigma_x \in \mathrm{SL}_2(\mathbf{Z})$ and $h \in \mathbf{Z}$ such that $\sigma_x \infty = x$ and

$$\sigma_x^{-1} \Gamma_x \sigma_x = \left\{ \begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\} \quad \text{or} \quad \sigma_x \Gamma_x \sigma_x^{-1} = \left\{ \pm \begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\}.$$

(Hint: consider first the case $x = \infty$.)

Solution:

Let $x \in \mathbf{P}^1(\mathbf{Q})$. By part a) we know that there is a $\sigma_x \in \mathrm{SL}_2(\mathbf{Z})$ for which $\sigma_x \cdot \infty = x$.

Note that for all $g \in \Gamma_x$ we have that

$$(\sigma_x^{-1} g \sigma_x) \cdot \infty = (\sigma_x^{-1} g) \cdot x = \infty,$$

so that $\sigma_x^{-1} \Gamma_x \sigma_x \subseteq \Gamma_\infty$. To obtain the other inclusion, note that for $\gamma \in \Gamma_\infty$, we have $\sigma_x^{-1} \gamma \sigma_x \in \Gamma_x$, so that $g = \sigma_x^{-1} \sigma_x \gamma \sigma_x^{-1} \sigma_x \in \sigma_x^{-1} \Gamma_x \sigma_x$, which implies the other inclusion as well.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \infty$$

if and only if $c = 0$, in which case $ad = \pm 1$. Hence the stabilizer of ∞ in $\mathrm{SL}_2(\mathbf{Z})$ is given by

$$\left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{Z} \right\}$$

Now note that $\Gamma_\infty < \mathrm{SL}_2(\mathbf{Z})_\infty$ is of finite index h, so that Γ_∞ must be of the form

$$\left\{ \pm \begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}$$

or

$$\left\{ \begin{pmatrix} 1 & hn \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}.$$

d) What are σ_x and h in the case of the unique cusp of $\mathrm{SL}_2(\mathbf{Z})$?
Solution:
By the explicit description we gave in part a), together with part c), where we have $h = 1$.

e) Let $f : \mathbf{H} \to \mathbf{C}$ be meromorphic and modular of weight k. Show that for every cusp x of Γ, there exists a function $\widetilde{f_x} : D^* \to \mathbf{C}$ such that

$$(f \mid_k \sigma_x)(z) = \widetilde{f_x}(e^{2i\pi z/h})$$

for $z \in \mathbf{H}$, where σ_x and h are as in Question c).
Solution:
Write $\sigma_x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. By part c) the matrix $\pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ is in Γ_∞, and also in Γ. Then

$$\begin{aligned}(f \mid_k \sigma_x)\left(\pm\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \cdot z\right) &= (cz+d)^{-k} f\left(\pm\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \cdot (\sigma_x \cdot z)\right) \\ &= (\pm 1)^k (cz+d)^{-k} f(\sigma_x \cdot z) \\ &= (\pm 1)^k (f \mid_k \sigma_x)(z)\end{aligned}$$

Hence if $g \in \Gamma$, then $f \mid_k \sigma_x$ is h-periodic, and moreover one can construct a function $\tilde{f}_x$ as it was done in the lectures for classical modular forms. If $-\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$, then the function $e^{-\pi z/h}(f \mid_k \sigma_x)$ is h-periodic in the sense that

$$e^{\pi(z+h)/h} = -e^{-\pi z/h},$$

so that we can construct a function $\tilde{f}$ such that $f(z) = \tilde{f}(e(z/h))$. Then we set $\tilde{f}_x = e^{\pi z/h}\tilde{f}(z)$ to obtain the required function.

One says that f is *holomorphic at the cusp* x if $\widetilde{f_x}$ is holomorphic at 0. The space of all modular forms of weight k for Γ which are holomorphic on $\mathbf{H}$ and at all cusps is denoted $M_k(\Gamma)$.

f) Check that $M_k(\mathrm{SL}_2(\mathbf{Z}))$ coincides with the space M_k of the lecture.
Solution:
Since the action of $\mathrm{SL}_2(\mathbf{Z})$ is transitive and hence only has only one orbit, it also has only one cusp. Thus it is enough to consider the cusp at ∞; the construction is then the same as the one that was done in the lectures. Then $M_k(\mathrm{SL}_2(\mathbf{Z})) = M_k$.

g) Let $C \subset \mathrm{SL}_2(\mathbf{Z})$ be a set of coset representatives of $\Gamma\backslash\mathrm{SL}_2(\mathbf{Z})$. Prove that if $f \in M_k(\Gamma)$, then

$$\prod_{g \in C} f \mid_k g$$

is an element of $M_{|C|k} = M_{|C|k}(\mathrm{SL}_2(\mathbf{Z}))$, which is non-zero if f is non-zero.
Solution:

Denote $G(z) = \prod_{g\in C}(f \mid_k g)(z)$ and $\gamma_k\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, z\right) = (cz+d)^{-k}$. Since f is holomorphic at infinity, it is bounded, so that the product remains bounded as well. Let $h \in \mathrm{SL}_2(\mathbf{Z})$. Then using the cocycle relation from Eercise sheet 1 for γ

$$\begin{aligned} G(h\cdot z) &= \prod_{g\in C}(f \mid_k g)(h\cdot z) \\ &= \prod_{g\in C}(c(h\cdot z)+d)^{-k} f(g\cdot(h\cdot z)) \\ &= \prod_{g\in C}\gamma_k(h,z)\gamma_{-k}(gh,z) f(g\cdot(h\cdot z)) \\ &= \gamma_k(h,z)^{|C|k}\prod_{g\in C}\gamma_{-k}(gh,z) f(g\cdot(h\cdot z)) \\ &= \gamma_k(h,z)^{|C|k}\prod_{g\in C}(f \mid_k g)(z) = \gamma_k(h,z)^{|C|k}G(z) \end{aligned}$$

Hence $G(z) \in M_{|C|k}$. From the expression above we see that G is non-zero if f is non-zero.

h) Deduce that there exists a constant $A_k \geq 0$ such that

$$\sum_{z\in\Gamma\backslash\mathbf{H}} \frac{v_z(f)}{e_z} \leq A_k$$

for all non-zero $f \in M_k(\Gamma)$, where $e_z = |\mathrm{Stab}_z(\Gamma)|$.

Solution:

For C as in part g) we have that $h(z)\prod_{g\in C}(f|_k g) \in M_k(\mathrm{SL}_2(\mathbf{Z}))$. Then by the $k/12$ (or the valence formula) from the lectures, we have

$$v_\infty(h) + \sum_{z\in\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \frac{v_z(h)}{e_z} = \frac{|C|k}{12},$$

from which we obtain

$$\sum_{z\in\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \frac{v_z(h)}{e_z} \leq \frac{|C|k}{12}.$$

By the multiplicativity of the valuation, note that

$$v_z(h) = \sum_{g\in C} v_z(f|_k g),$$

so that there exists a $g \in C$ such that

$$v_z(f) \leq |C| v_z(f|_k g),$$

so that alltogether

$$\sum_{z\in\Gamma\backslash\mathbf{H}} \frac{v_z(f)}{e_z} \leq |C| \sum_{z\in\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \frac{v_z(h)}{e_z} \leq \frac{|C|^2 k}{12} =: A_k$$

Alternative solution:
Let $f \in M_k(\Gamma)$ be non-zero. From the valence formula in the lectures note that we have

$$\sum_{z \in \Gamma \backslash \mathbf{H}} \frac{v_z(f)}{|\mathrm{Stab}_z(\overline{\Gamma})|} + \sum_{P \in \mathrm{Cusps}(\Gamma)} v_P(f) = \frac{k[\mathrm{PSL}_2(\mathbf{Z}) : \overline{\Gamma}]}{12},$$

so by setting $A_k := \frac{k[\mathrm{PSL}_2(\mathbf{Z}):\overline{\Gamma}]}{12}$, we have $\sum_{z \in \Gamma \backslash \mathbf{H}} v_z(f) \leq A_k$.

i) Prove that $\dim M_k(\Gamma) \leq A_k + 1$.
Solution:
Let $z_0 \in \mathbf{C}$. With A_k as in the solution of the previous exercise, consider the linear map $M_k(\Gamma) \to \mathbf{C}^{A_k+1}$, mapping f to the coefficients $(f(z_0), f'(z_0), \ldots.., f^{(A_k)}(z_0))$. We claim that this map is injective. Let f and g be modular forms of weight k and level Γ, such that their q-expansions agree up to degree $\lfloor A_k \rfloor$. We will show that then $f = g$. Note that then

$$v_\infty(f-g) \geq 1 + \lfloor A_k \rfloor > A_k.$$

This yields a contradiction to the valence formula (stated in part h)), unless $f - g = 0$. This proves the injectivity of the map. Hence $\dim M_k(\Gamma) \leq A_k + 1$.

3. Let $k \geq 2$ be an even integer. Recall that the Bessel function J_k is defined for $z \in \mathbf{C}$ by

$$J_k(z) = \sum_{n \geq 0} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{k+2n}.$$

a) Prove that for all $z \in \mathbf{C}$, we have

$$J_k(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin(\vartheta) - ik\vartheta} d\vartheta,$$

and deduce that $|J_n(x)| \leq 1$ for all $x \in \mathbf{R}$.
Solution:
We start by writing out the power series expansion for the exponential $e^{iz\sin(\vartheta)}$ explicitly, followed by writing out the binomial expansion of $\sin(\vartheta) = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$ as follows

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin(\vartheta)-ik\vartheta}d\vartheta &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta}\sum_{n=0}^{\infty}\frac{1}{n!}(iz)^n\sin(\vartheta)^n d\vartheta \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(iz)^n\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta}\sin(\vartheta)^n d\vartheta \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(iz)^n\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta}\left(\frac{e^{i\vartheta}-e^{-i\vartheta}}{2i}\right)^n d\vartheta \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta}\sum_{l=0}^{n}\binom{n}{l}(-1)^l e^{i\vartheta(n-l)}e^{-i\vartheta l}d\vartheta \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta}e^{i\vartheta(n-l)}e^{-i\vartheta l}d\vartheta \\
&= \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\sum_{l=0}^{n}\frac{1}{l!(n-l)!}(-1)^l\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta+i\vartheta(n-l)-i\vartheta l}d\vartheta
\end{aligned}
$$

For any integer a the integral $\int_0^{2\pi} e^{ia\vartheta}d\vartheta$ is non-zero if and only if $a=0$. Hence only for $n=k+2l$ the integral above will be non-zero. Moreover

$$
\frac{1}{2\pi}\int_0^{2\pi} d\vartheta = 1.
$$

Thus

$$
\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\sum_{l=0}^{n}\frac{1}{l!(n-l)!}(-1)^l\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\vartheta+i\vartheta(n-l)-i\vartheta l}d\vartheta = \sum_{l=0}^{\infty}\left(\frac{z}{2}\right)^{k+2l}\frac{1}{l!(k+2l-l)!}(-1)^l
$$

Let $x \in \mathbf{R}$. Then $|J_n(x)| \leq 1$ follows from the inequality

$$
\begin{aligned}
|J_k(x)| &\leq \frac{1}{2\pi}\int_0^{2\pi} |e^{i(x\sin(\vartheta)-k\vartheta)}|d\vartheta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\cos(x\sin(\vartheta)-k\vartheta)+i\sin(x\sin(\vartheta)-k\vartheta)|d\vartheta \\
&= 1
\end{aligned}
$$

b) Prove that the function $f(z)=J_n(z)$ satisfies the differential equation

$$
z^2 f''(z) + zf'(z) + (z^2-n^2)f(z) = 0.
$$

Solution:

Taking the derivative of $J_n(z)$:

$$
J_n'(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k!\,(n+k)!}\cdot(2k+n)\left(\frac{z}{2}\right)^{2k+n-1}
$$

Multiplying by z, we obtain:

$$zJ_n'(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \cdot (2k+n) \left(\frac{z}{2}\right)^{2k+n}$$

Taking the derivative again:

$$J_n''(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4k!\,(n+k)!} \cdot (2k+n)(2k+n-1) \left(\frac{z}{2}\right)^{2k+n-2}$$

Multiplying by z^2, we get:

$$z^2 J_n''(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \cdot (2k+n)(2k+n-1) \left(\frac{z}{2}\right)^{2k+n}$$

Then

$$z^2 J_n''(z) + zJ_n' - n^2 J_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{z}{2}\right)^{2k+n} \cdot 4k(k+n).$$

On the other hand

$$\begin{aligned} z^2 J_n(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{z}{2}\right)^{2(k+1)+n} \cdot 4 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!\,(n+k+1)!} \left(\frac{z}{2}\right)^{2(k+1)+n} \cdot (-4(k+1)(n+k+1)) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(k)!\,(n+k)!} \left(\frac{z}{2}\right)^{2k+n} \cdot (-4k(n+k)) \end{aligned}$$

Hence we are only left with the 0-term in the sum over k:

$$z^2 J_n''(z) + zJ_n'(z) + (z^2 - n^2) J_n(z) = \frac{1}{n!} \left(\frac{z}{2}\right)^n \cdot 4 \cdot 0 \cdot n = 0.$$

Thus, the Bessel function satisfies the differential equation.

4. Let p be a prime number. Recall that the Kloosterman sum $S(m,n;p)$ is defined by

$$S(m,n;p) = \sum_{x \in \mathbf{F}_p^\times} e\Big(\frac{mx + n\bar{x}}{p}\Big), \qquad x\bar{x} \equiv 1 \bmod p.$$

a) Show that $S(m,n;p) \in \mathbf{R}$.

Solution:

We have to show that $S(m,n;p)$ doesn't change under complex-conjugation. Note that

$$
\begin{aligned}
\overline{S(m,n;p)} &= \sum_{x\in\mathbf{F}_p^\times} e\Big(\frac{m(-x)+n(-\bar{x})}{p}\Big) \\
&= \sum_{x\in\mathbf{F}_p^\times} e\Big(\frac{m(-x)+n(\overline{-x})}{p}\Big), \qquad \text{as } (-x)(\overline{-x}) \equiv (-x)(-\bar{x}) \equiv 1 \pmod p \\
&= \sum_{-x\in\mathbf{F}_p^\times} e\Big(\frac{mx+n\bar{x}}{p}\Big) = S(m,n;p).
\end{aligned}
$$

b) Show that

$$\sum_{m,n\in\mathbf{F}_p} |S(m,n;p)|^4 = p^2 N(p)$$

where $N(p)$ is the number of solutions $(a,b,c,d) \in (\mathbf{F}_p^\times)^4$ of the equations

$$\begin{cases} a+b = c+d \\ a^{-1}+b^{-1} = c^{-1}+d^{-1}. \end{cases}$$

Solution:

For h an integer we will use the formula

$$\frac{1}{p}\sum_{m\in\mathbf{F}_p} e\left(\frac{mh}{p}\right) = \delta_{0,h} = \begin{cases} 1, & h=0 \\ 0, & else. \end{cases} \tag{1}$$

We compute the fourth power:

$$
\begin{aligned}
\sum_{m,n\in\mathbf{F}_p} |S(m,n;p)|^4 &= \sum_{m,n\in\mathbf{F}_p} \sum_{a,b,c,d\in(\mathbf{F}_p)^\times} e\left(\frac{m(a+b-c-d)}{p}\right) e\left(\frac{n(a^{-1}+b^{-1}-c^{-1}-d^{-1})}{p}\right) \\
&= \sum_{a,b,c,d\in(\mathbf{F}_p)^\times} \left(\sum_{m\in\mathbf{F}_p} e\left(\frac{m(a+b-c-d)}{p}\right)\right)\left(\sum_{n\in\mathbf{F}_p} e\left(\frac{n(a^{-1}+b^{-1}-c^{-1}-d^{-1})}{p}\right)\right) \\
&= p^2 \sum_{\substack{(a,b,c,d)\in((\mathbf{F}_p)^\times)^4 \\ a+b=c+d \\ a^{-1}+b^{-1}=c^{-1}+d^{-1}}} 1 = p^2 N(p),
\end{aligned}
$$

where when going from the second to the third row we used the formula (1).

c) For $(m_0,n_0) \in (\mathbf{F}_p^\times)^2$, prove that

$$(p-1)|S(m_0,n_0;p)|^4 \le \sum_{m,n\in\mathbf{F}_p} |S(m,n;p)|^4.$$

Solution:
Let $a, m_0, n_0 \in \mathbf{F}_p^\times$ be arbitrary. Then

$$\begin{aligned}
S(am_0, a^{-1}n_0; p) &= \sum_{x \in \mathbf{F}_p^\times} e\Big(\frac{m_0 ax + n_0 a^{-1}\bar{x}}{p}\Big), \qquad x\bar{x} \equiv 1 \bmod p \\
&= \sum_{ax \in \mathbf{F}_p^\times} e\Big(\frac{m_0 ax + n_0 \overline{ax}}{p}\Big) \\
&= S(m_0, n_0; p).
\end{aligned}$$

Let $\tilde{m} \in \mathbf{F}_p^\times$ be such that $|S(\tilde{m}, n; p)|$ is minimal under all $|S(m, n; p)|$, for $m \in \mathbf{F}_p^\times$.
Let $a \in \mathbf{F}_p^\times$ be such that $a\tilde{m} = m_0$. Then

$$\begin{aligned}
\sum_{m,n \in \mathbf{F}_p} |S(m, n; p)|^4 &\geq \sum_{n \in \mathbf{F}_p} \sum_{m \in \mathbf{F}_p^\times} |S(\tilde{m}, n; p)|^4 \\
&= \sum_{n \in \mathbf{F}_p} (p-1)|S(\tilde{m}, n; p)|^4 \\
&= \sum_{n \in \mathbf{F}_p} (p-1)|S(a\tilde{m}, a^{-1}n; p)|^4 \\
&= \sum_{n \in \mathbf{F}_p} (p-1)|S(m_0, a^{-1}n; p)|^4 \\
&\geq (p-1)|S(m_0, n_0; p)|^4.
\end{aligned}$$

d) Deduce that for $(m_0, n_0) \in (\mathbf{F}_p^\times)^2$, we have

$$S(m_0, n_0; p) = O(p^{3/4}).$$

Solution:
Let $m_0, n_0 \in \mathbf{F}_p^\times$ be arbitrary.
It suffices to prove that $N(p) = O(p^2)$, since then by part b) we obtain

$$\sum_{m,n \in \mathbf{F}_p} |S(m, n; p)|^4 = p^2 N(p) = O(p^4).$$

Then from part c) it follows that

$$|S(m_0, n_0; p)|^4 = O(p^3),$$

so that $S(m_0, n_0; p) = O(p^{3/4})$.
Now we will prove $N(p) = O(p^2)$. First fix values $a, b \in \mathbf{F}_p^\times$ with $a + b \neq 0$. If $(c, d) \in (\mathbf{F}_p^\times)^2$ satisfy

$$a + b = c + d \tag{2}$$
$$a^{-1} + b^{-1} = c^{-1} + d^{-1}, \tag{3}$$

then the value of $c+d$ is fixed and $a^{-1}+b^{-1}\neq 0$. Hence

$$cd = \frac{c+d}{a^{-1}+b^{-1}} \in \mathbf{F}_p^\times$$

is fixed. Since we know both $c+d$ and cd, there are at most 2 pairs (c,d) that satisfy (2) and (3). Hence the number of choices for a,b,c,d in this case is bounded by a constant times p^2.

If $a+b=0$, then $c+d=0$, so that the solution is determined by (a,c), which in total leves us with p^2 choices. Taking both cases into account, we obtain our claim.