Solutions: Exercise Sheet 4

1. Let $q \geq 1$ be an integer and let $f \in S_k(q)$ be a cusp form of weight k for $\Gamma_0(q)$ (with trivial nebentypus). Let (a_n) be the Fourier coefficients of f at ∞.

Let $\alpha \in \mathbf{R}$.

a) Show that for any $y > 0$ and integer $N \geq 1$, we have

$$\sum_{n \leq N} a_n e(n\alpha) = \int_0^1 f(t + iy + \alpha)\Big(\sum_{1 \leq n \leq N} e(-n(t + iy))\Big)dt.$$

Solution:
Let $\alpha \in \mathbf{R}$. Write $z = x + iy$, with $x, y \in \mathbf{R}$. If we write $f(z) = \sum a_n e^{2\pi i n z}$, then

$$a_n = \int_0^1 f(z)e(-nz)dx.$$

Then the expression on the left in the equation of part a) is

$$\begin{aligned}\sum_{n \leq N} a_n e(n\alpha) &= \sum_{n \leq N} \int_0^1 f(z)e(-nz + n\alpha)dx \\ &= \int_0^1 f(x + iy) \left(\sum_{n \leq N} e(-n(x + iy - \alpha))\right) dx \\ &= \int_0^1 f(t + iy + \alpha) \left(\sum_{n \leq N} e(-n(t + iy))\right) dt\end{aligned}$$

where in the last line we applied a change of coordinates $t = x + \alpha$.

b) Deduce that for $N \geq 1$ and $\alpha \in \mathbf{R}$, we have

$$\sum_{n \leq N} a_n e(n\alpha) = O(N^{k/2} \log N),$$

where the implied constant depends only on f.

Solution:
Consider the geometric sum

$$\sum_{1 \leq n \leq N} e(-nz) = \frac{e(-Nz) - 1}{1 - e(z)},$$

From it we obtain the upper bound

$$\frac{e(-Nz)-1}{1-e(z)} \ll e^{2\pi Ny}|1-e(z)|^{-1},$$

Writing $z = x + iy$, note that $f(z+\alpha) \ll y^{-k/2}$ (by either a lemma from the lectures or Exercise 1 from Exercise Sheet 2). We further have

$$\int_0^1 |1-e(z)|^{-1}dx \ll \log(2+y^{-1}).$$

Hence

$$\begin{aligned}
\sum_{n\leq N} a_n e(n\alpha) &= \int_0^1 f(t+iy+\alpha)\left(\sum_{n\leq N} e(-n(t+iy))\right)dt \\
&= \int_0^1 f(t+iy+\alpha)\frac{e(-N(t+iy))-1}{1-e(t+iy)}dt \\
&\ll y^{-k/2}e^{2\pi Ny}\int_0^1 |1-e(t+iy)|^{-1}dt \\
&\ll y^{-k/2}e^{2\pi Ny}\log(2+y^{-1}),
\end{aligned}$$

for y an arbitrary positive real number. Setting $y = N^{-1}$ proves the upper bound.

2. Let $q \geq 1$ be a prime number and let $f \in S_k(q)$ be a cusp form of weight k for $\Gamma_0(q)$ (with trivial nebentypus). Let

$$f(z) = \sum_{n\geq 1} a_n e(nz)$$

be the Fourier expansion of f at ∞.

Let $r \geq 1$ be a prime number distinct from q and let χ be a non-trivial Dirichlet character modulo r. Define

$$(f\times\chi)(z) = \sum_{n\geq 1} a_n\chi(n)e(nz)$$

for $z \in \mathbf{H}$.

a) Show that the Gauss sum

$$\tau(\chi) = \sum_{x\in\mathbf{Z}/r\mathbf{Z}} \chi(x)e(x/r)$$

satisfies $|\tau(\chi)| = \sqrt{r}$.

Solution:

Compute

$$\begin{aligned}
|\tau(\chi)|^2 &= \tau(\chi)\overline{\tau(\chi)} \\
&= \sum_{a\in\mathbf{Z}/r\mathbf{Z}} \chi(a) \sum_{b\in\mathbf{Z}/r\mathbf{Z}} \overline{\chi}(b) e\left(\frac{a-b}{r}\right) \\
&= \sum_{c\in\mathbf{Z}/r\mathbf{Z}} \chi(c) \sum_{b\in\mathbf{Z}/r\mathbf{Z}} e\left(\frac{(c-1)b}{r}\right),
\end{aligned}$$

where $c = ab^{-1}$. Then by the orthogonality of the character, we have

$$\sum_{b\in\mathbf{Z}/r\mathbf{Z}} e\left(\frac{(c-1)b}{r}\right) = r\delta_{0,c-1},$$

so that

$$\sum_{c\in\mathbf{Z}/r\mathbf{Z}} \chi(c) \sum_{b\in\mathbf{Z}/b\mathbf{Z}} e\left(\frac{(c-1)b}{r}\right) = r.$$

Taking the square-root we are done.

b) Show that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{x\in\mathbf{Z}/r\mathbf{Z}} \overline{\chi}(x) e(nx/r)$$

for all integers $n \geq 1$.

Solution:

We have

$$\begin{aligned}
\sum_{x\in\mathbf{Z}/r\mathbf{Z}} \chi(x)e(nx/r) &= \sum_{x\in\mathbf{Z}/r\mathbf{Z}} \chi(nx)\chi(n^{-1})e(nx/r) \\
&= \overline{\chi}(n) \sum_{x\in\mathbf{Z}/r\mathbf{Z}} \chi(nx) e\left(\frac{nx}{r}\right) \\
&= \overline{\chi}(n)\tau(\chi)
\end{aligned}$$

c) Show that

$$(f\times\chi)(z) = \frac{1}{\tau(\overline{\chi})} \sum_{0\leq u\leq r-1} \overline{\chi}(u) f\Big(z+\frac{u}{r}\Big).$$

Solution:

Using part b) we can compute

$$\begin{aligned}
(f\times\chi)(z) &= \sum_{n\geq 1} a_n \chi(n) e(nz) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{n\geq 1} a_n \left(\sum_{u\in\mathbf{Z}/r\mathbf{Z}} \overline{\chi}(u) e(nu/r) \right) e(nz) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{u\in\mathbf{Z}/r\mathbf{Z}} \overline{\chi}(u) \sum_{n\geq 1} a_n e\left(n\left(z+\frac{u}{r}\right)\right)
\end{aligned}$$

d) Show that

$$((f \times \chi) \mid_k g)(z) = \chi(d)^2 (f \times \chi)(z)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(qr^2)$.

Solution:

Consider the identity

$$\begin{pmatrix} 1 & \frac{u}{r} \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\frac{d^2u}{r} \\ & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{uc}{r} & b - \frac{bcdu}{r} - \frac{cd^2u^2}{r^2} \\ c & d - \frac{cd^2u}{r} \end{pmatrix} =: A$$

So if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(qr^2)$, then $A \in \Gamma_0(qr^2)$ and moreover $A \equiv \begin{pmatrix} a & * \\ & d \end{pmatrix}$ (mod q). Then $\det(A) = ad - bc = 1$, so that in particular $A \in \Gamma_0(q)$ (since $gr^2 \mid c$). Hence we have $f|_k A = f$. Thus

$$\begin{aligned}
((f \times \chi) \mid_k g)(z) &= \frac{1}{\tau(\overline{\chi})} \sum_{0 \le u \le r-1} \overline{\chi}(u)(f|_k g)\left(z + \frac{u}{r}\right) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{0 \le u \le r-1} \overline{\chi}(u) \left(f|_k \begin{pmatrix} 1 & u/r \\ & 1 \end{pmatrix} g\right)(z) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{0 \le u \le r-1} \overline{\chi}(u) \left((f|_k A) \begin{pmatrix} 1 & d^2u/r \\ & 1 \end{pmatrix}\right)(z) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{0 \le u \le r-1} \chi(d^2)\overline{\chi}(d^2u) f\left(z + \frac{d^2u}{r}\right) \\
&= \frac{\chi^2(d)}{\tau(\overline{\chi})} \sum_{0 \le u \le r-1} \overline{\chi}(u) f\left(z + \frac{u}{r}\right) \\
&= \chi(d)^2 (f \times \chi)(z)
\end{aligned}$$

e) Deduce that $f \times \chi \in S_k(qr^2, \chi^2)$.

Solution:

By part d) we obtain the automorphy of $f \times \chi$. We are left to check that $f \times \chi$ does not grow too fast at the boundary of $\mathbf{H}$. By part c) we have that

$$\operatorname{Im}(z)^{\frac{k}{2}} |(f \times \chi)(z)| \le \frac{1}{\sqrt{r}} \sum_{0 \le u \le r-1} |\chi(u)| \operatorname{Im}(z)^{\frac{k}{2}} |f(z + u/r)|, \qquad (1)$$

and since f is a cusp form, we have that $\operatorname{Im}(z)^{\frac{k}{2}} |f(z + u/r)|$ is bounded on $\mathbf{H}$, so the expression on the left hand side of equation (1) is bounded as well, and by Exercise 1c) of Exercise sheet 2, we have that $f \times \chi$ is a cusp form.

3. Let $q \ge 1$ be an integer and let $f \in S_k(q)$ be a cusp form of weight k for $\Gamma_0(q)$ (with trivial nebentypus).

a) For an integer $d \geq 1$, show that

$$g(z) = f(dz)$$

defines a cusp form $g \in S_k(dq)$.

Solution:

From the definition of g we see that g is holomorphic on the upper-half plane, together with ∞. Since f is a cusp form we are left to check that g satisfies the automorphy transformation property. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(dq)$. Then $d \mid C$, so that both matrices $\begin{pmatrix} A & Bd \\ C/d & D \end{pmatrix}$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are in $\Gamma_0(q)$.

From the identity

$$\begin{pmatrix} d & \\ & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & Bd \\ C/d & D \end{pmatrix} \begin{pmatrix} d & \\ & 1 \end{pmatrix}$$

it follows that $g\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot z\right) = (Cz + D)^k g(z)$. More precisely, we compute

$$\begin{aligned} g\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot z\right) &= f\left(d\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot z\right) \\ &= f\left(\begin{pmatrix} A & Bd \\ C/d & D \end{pmatrix} \cdot (dz)\right) \\ &= (Cz + D)^k f(dz) \end{aligned}$$

b) Show that if $m \geq 1$ is an integer coprime to dq, we have

$$(T(m)g)(z) = (T(m)f)(dz),$$

where $T(m)$ is the m-th Hecke operator.

Solution:

From the definitions we obtain

$$\begin{aligned} (T(m)g)(z) &= m^{k-1} \sum_{a|m} \sum_{b=0}^{\frac{m}{a}-1} g\left(\frac{az+b}{m/a}\right)\left(\frac{a}{m/a}\right)^{\frac{k}{2}} \\ &= m^{k-1} \sum_{a|m} \sum_{b=0}^{\frac{m}{a}-1} f\left(d\frac{az+b}{m/a}\right)\left(\frac{a}{m/a}\right)^{\frac{k}{2}} \\ &= m^{k-1} \sum_{a|m} \sum_{c}^{\frac{m}{a}-1} f\left(\frac{a(dz)+c}{m/a}\right)\left(\frac{a}{m/a}\right)^{\frac{k}{2}} \end{aligned}$$

where the last sum is taken over c such that $c \equiv db \pmod{m/a}$