Exercise Sheet 2

1. a) Show that the measure

$$\mu = \frac{dxdy}{y^2}$$

on $\mathbf{H}$ (with coordinate $z = x + iy$) is invariant under the action of $\mathrm{SL}_2(\mathbf{R})$: for any $g \in \mathrm{SL}_2(\mathbf{R})$ and any μ-integrable function $f\colon \mathbf{H} \to \mathbf{C}$, we have

$$\int_{\mathbf{H}} f(g \cdot z) d\mu(z) = \int_{\mathbf{H}} f(z) d\mu(z).$$

Solution:
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ be arbitrary, and let $f : \mathbf{H} \to \mathbf{C}$ be a μ-integrable function. For $z = x_1 + iy_1$ we will write $A \circ z = A(x_1 + iy_1) = x_2 + iy_2$, for $x_i, y_i \in \mathbf{R}$. Note that

$$y_2 = \frac{1}{|cz+d|^2} \left(ay_1(cx_1 + d) - cy_1(ax_1 + b) \right) = \frac{y_1}{|cz+d|^2}$$

Then

$$\begin{aligned}
\int_{\mathbf{H}} f(x_2 + iy_2) \frac{dx_2 dy_2}{y_2^2} &= \int_{\mathbf{H}} f(x_2 + iy_2) \left(\frac{1}{|z+d|^2} \right)^2 \frac{|cz+d|^4}{y_2^2} dx_2 dy_2 \\
&= \int_{\mathbf{H}} f(x_2 + iy_2) \left| \frac{dA}{dz} \right|^2 \left(\frac{|cz+d|^2}{y_2} \right)^2 dx_2 dy_2 \\
&= \int_{\mathbf{H}} f(A(x_1 + iy_1)) \left| \frac{dA}{dz} \right|^2 \frac{dx_2 dy_2}{y_1^2} \\
&= \int_{A(\mathbf{H})} f\left(A^{-1}\left(A(x_1 + iy_1) \right) \right) \frac{dx_1 dy_1}{y_1^2} = \int_{\mathbf{H}} f(z) \frac{dx_1 dy_1}{y_1^2},
\end{aligned}$$

where we used the Cauchy-Riemann equations in the last line.

b) Let $f\colon \mathbf{H} \to \mathbf{C}$ be any function which is modular of weight $k \in \mathbf{Z}$. Show that the function ϕ defined on $\mathbf{H}$ by

$$\phi(z) = |f(z)| \mathrm{Im}(z)^{k/2}$$

is modular of weight 0 (i.e., is an $\mathrm{SL}_2(\mathbf{Z})$-invariant function on $\mathbf{H}$).

Solution:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ be arbitrary, and let $f : \mathbf{H} \to \mathbf{C}$ be modular of weight k. Then

$$\begin{aligned}\phi(A \circ z) &= |f(A \circ z)| \mathrm{Im}(A \circ z)^{k/2} \\ &= |cz+d|^k |f(z)| \left(\frac{\mathrm{Im}(z)}{|cz+d|^2} \right)^{k/2} = \phi(z).\end{aligned}$$

c) Suppose that f is furthermore meromorphic on $\mathbf{H}$ and modular of weight $k \geq 2$. Show that f is a cusp form if and only if ϕ is bounded on $\mathbf{H}$.
Solution:
Let f be a cusp form. By modularity it is enough to consider the behavior on the fundamental domain $\mathcal{F}$. We need to show that $\phi(z)$ is bounded on $\mathrm{Im}(z) \geq h$, for any $h > 0$. Since f be a cusp form, it is holomorphic and $\lim_{\mathrm{Im(z)} \to \infty} |f(z)| = 0$. Write $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ and $z = x + yi$. Then

$$f(x+yi) = e^{-2\pi y} \sum_{n=1}^{\infty} a_n e^{2\pi i n x} e^{-2\pi(n-1)y},$$

and since f is homolorphic, the sum on the right si finite. Hence there exists a positive constant $\tilde{C}$ such that

$$|f(x+yi)| \leq e^{-2\pi y} \tilde{C}.$$

Hence there exists a positive constant $C > 0$ such that

$$|\phi(z)| = |f(z)| \mathrm{Im}(z)^{\frac{k}{2}} \leq e^{-2\pi y} \tilde{C} \mathrm{Im}(z)^{\frac{k}{2}} < C$$

Next we will prove the other direction. Assume that ϕ is bounded on $\mathbf{H}$.
Say that f is not holomorphic at ∞. Then f has a pole at ∞, and $\lim_{\mathrm{Im}(z) \to \infty} f(z) = \infty$. Hence $\phi(z)$ is not bounded; contradiction.
Let $\tilde{f}(z) = f(e^{2\pi i z})$. Assume that $\tilde{f}(0) \neq 0$ (so $\tilde{f}$ is not a cusp form), then

$$\phi(0) = \lim_{\mathrm{Im}(z) \to \infty} \tilde{f}(z) \mathrm{Im}(z)^{\frac{k}{2}}$$

is not bounded as $\tilde{f}(z) \neq 0$, but $\mathrm{Im}(z)^{\frac{k}{2}} \to \infty$; contradiction.

2. The goal of this exercise is to prove that the function Δ defined by

$$\Delta(z) = e(z) \prod_{n \geq 1} (1 - e(nz))^{24}$$

for $z \in \mathbf{H}$ is a cusp form of weight 12, where we recall that $e(z) = e^{2i\pi z}$ for $z \in \mathbf{C}$.

For $z \in \mathbf{C}$ with $\sin(z) \neq 0$, we define

$$\mathrm{cotan}(z) = \frac{\cos(z)}{\sin(z)}.$$

We fix a complex number $\tau \in \mathbf{H}$.

a) Prove that the infinite product converges locally uniformly absolutely, and hence that Δ is a well-defined holomorphic function on $\mathbf{H}$.

Solution:

Note that it is enough to prove absolute local uniform convergence for the term $\prod_{n\geq 1}(1-e(nz))$. We need to show that the sum $\sum_{n\geq 1}|e(nz)|$ converges absolutely locally uniformly. It is enough to show uniform convergence on

$$\{z \in \mathrm{H} : \mathrm{Im}(z) \geq d\},$$

for some $d > 0$. Compute

$$\sum_{n\geq 1}|e(nz)| = \sum_{n\geq 1}|e^{2\pi i n z}| = \sum_{n\geq 1} e^{-2\pi n \mathrm{Im}(z)} \leq \sum_{n\geq 1} e^{-2\pi n d},$$

which is a finite sum that converges absolutely and uniformly in z.

b) Show that cotan defines a meromorphic function on $\mathbf{C}$ with simple poles at $z = k\pi$ for $k \in \mathbf{Z}$ with residue 1. Prove that

$$\mathrm{cotan}(z) = -i\Big(1 - \frac{2}{1-e^{-2iz}}\Big)$$

for $z \in \mathbf{C}$.

Solution:

Since $\mathrm{cotan}(z) = \cos(z)/\sin(z)$, cotan is a meromorphic function (because both cos and sin are holomorphic). The poles of cotan are given by zeros of sin, which are at $z = k\pi$ and are all simple. Since $\cos(k\pi) \neq 0$, it follows that cotan has simple poles at $z = k\pi$, with $k \in \mathbf{Z}$.

The residue is given by

$$\mathrm{Res}_{k\pi}\,\mathrm{cotan}(z) = \lim_{z\to k\pi}(z-k\pi)\,\mathrm{cotan}(z) = \lim_{\omega\to 0}\frac{\cos(\omega)}{\frac{\sin(\omega)}{\omega}} = 1.$$

To obtain the formula for the cotangent, compute

$$\mathrm{cotan}(z) = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = i\frac{1+e^{-2iz}}{1-e^{-2iz}} = -i\Big(1 - \frac{2}{1-e^{-2iz}}\Big).$$

c) Let $m \geq 0$ be an integer and define meromorphic functions f_m and g_m by

$$f_m(z) = \mathrm{cotan}((m+\tfrac{1}{2})z)\,\mathrm{cotan}((m+\tfrac{1}{2})z/\tau)$$

and $g_m(z) = z^{-1}f_m(z)$. Show that g_m has

i) simple poles at $\pi k/(m+\frac{1}{2})$ for $k \in \mathbf{Z}$, k non-zero;

ii) simple poles at $\pi k\tau/(m+\frac{1}{2})$ for $k \in \mathbf{Z}$, k non-zero integer;

iii) a triple pole at $z = 0$.

Solution:

i) Note that

$$g_m(z) = \frac{1}{z}\frac{\cos((m+\frac{1}{2})z)}{\sin((m+\frac{1}{2})z)}\frac{\cos((m+\frac{1}{2})z/\tau)}{\sin((m+\frac{1}{2})z/\tau)}.$$

for $z = \frac{k\pi}{m+\frac{1}{2}}$, the factor $\frac{\cos((m+\frac{1}{2})z)}{\sin((m+\frac{1}{2})z)}$ has a simple pole, while the factor $\frac{\cos((m+\frac{1}{2})z/\tau)}{\sin((m+\frac{1}{2})z\tau)}$ is non-zero. Hence g_m has a simple pole at z.

ii) The argument is completely analogous to part (i); where we take $z = \frac{k\pi}{m+\frac{1}{2}}\tau$.

iii) The function cotan has a simple pole at 0, just like $z \mapsto 1/z$. Hence g_m has a triple pole, by additivity of the valuation.

d) Show that

$$\operatorname{Res}_{z=\pi k/(m+\frac{1}{2})} g_m(z) = \frac{1}{\pi k}\operatorname{cotan}(\pi k/\tau), \qquad \operatorname{Res}_{z=\pi k\tau/(m+\frac{1}{2})} g_m(z) = \frac{1}{\pi k}\operatorname{cotan}(\pi k\tau),$$

and

$$\operatorname{Res}_{z=0} g_m(z) = -\frac{1}{3}(\tau+\tau^{-1}).$$

Solution:
Compute:

$$\operatorname{Res}_{z=\pi k/(m+\frac{1}{2})} g_m(z) = \lim_{z\to\pi k/(m+\frac{1}{2})}\left(\left(z-\frac{k\pi}{m+\frac{1}{2}}\right)\frac{1}{z}\operatorname{cotan}((m+\tfrac{1}{2})z)\operatorname{cotan}((m+\tfrac{1}{2})z/\tau)\right)$$

$$= \lim_{\omega\to 0}\left(\left(\frac{1}{\omega(m+\frac{1}{2})+k\pi}\right)\cdot\omega(m+\tfrac{1}{2})\operatorname{cotan}(\omega(m+\tfrac{1}{2}))\cdot\operatorname{cotan}\left(\frac{\omega(m+\frac{1}{2})}{\tau}+\frac{k\pi}{\tau}\right)\right),$$

where we have set $\omega := \frac{z(m+\frac{1}{2})-k\pi}{m+\frac{1}{2}}$. Since

$$\lim_{\omega\to 0}\left(\frac{1}{\omega(m+\frac{1}{2})+k\pi}\right) = \frac{1}{k\pi}$$

$$\lim_{\omega\to 0}\left(\omega(m+\tfrac{1}{2})\operatorname{cotan}(\omega(m+\tfrac{1}{2}))\right) = 1$$

$$\lim_{\omega\to 0}\left(\operatorname{cotan}\left(\frac{\omega(m+\frac{1}{2})}{\tau}+\frac{k\pi}{\tau}\right)\right) = \operatorname{cotan}(k\pi/\tau),$$

we obtain that

$$\operatorname{Res}_{z=\pi k/(m+\frac{1}{2})} g_m(z) = \frac{1}{k\pi}\operatorname{cotan}(k\pi/\tau).$$

The next residue is computed similarly:

$$\operatorname{Res}_{z=\pi k\tau/(m+\frac{1}{2})} g_m(z) = \lim_{z\to\pi k/(m+\frac{1}{2})}\left(\left(z-\frac{k\pi\tau}{m+\frac{1}{2}}\right)\frac{1}{z}\operatorname{cotan}((m+\tfrac{1}{2})z)\operatorname{cotan}((m+\tfrac{1}{2})z/\tau)\right)$$

$$= \lim_{\omega\to 0}\left(\left(\frac{1}{\omega(m+\frac{1}{2})+k\pi\tau}\right)\omega(m+\tfrac{1}{2})\operatorname{cotan}(\omega(m+\tfrac{1}{2})+k\pi\tau)\operatorname{cotan}\left(\frac{\omega(m+\frac{1}{2})}{\tau}\right)\right)$$

$$= \lim_{\omega\to 0}\left(\left(\frac{\tau}{\omega(m+\frac{1}{2})+k\pi\tau}\right)\cdot\operatorname{cotan}(\omega(m+\tfrac{1}{2})+k\pi\tau)\cdot\frac{\omega(m+\frac{1}{2})}{\tau}\operatorname{cotan}\left(\frac{\omega(m+\frac{1}{2})}{\tau}\right)\right),$$

where we have set $\omega := \frac{z(m+\frac{1}{2})-k\pi\tau}{m+\frac{1}{2}}$. Since

$$\lim_{\omega\to 0}\left(\frac{\tau}{\omega(m+\frac{1}{2})+k\pi\tau}\right)=\frac{1}{k\pi}$$
$$\lim_{\omega\to 0}\left(\frac{\omega(m+\frac{1}{2})}{\tau}\operatorname{cotan}\left(\frac{\omega(m+\frac{1}{2})}{\tau}\right)\right)=1$$
$$\lim_{\omega\to 0}\left(\operatorname{cotan}\left(\omega(m+\tfrac{1}{2})+k\pi\tau\right)\right)=\operatorname{cotan}(k\pi\tau),$$

we obtain that

$$\operatorname{Res}_{z=\pi k\tau/(m+\frac{1}{2})} g_m(z)=\frac{1}{\pi k}\operatorname{cotan}(\pi k\tau).$$

The Laurent expansion of $\operatorname{cotan}(z)$ near $z=0$ is

$$\operatorname{cotan}(z)=\frac{1}{z}-\frac{z}{3}+O(z^3).$$

Applying this to $f_m(z)$, we get

$$\operatorname{cotan}\left((m+\tfrac{1}{2})z\right)=\frac{1}{(m+\frac{1}{2})z}-\frac{(m+\frac{1}{2})z}{3}+O(z^3),$$
$$\operatorname{cotan}\left(\frac{(m+\frac{1}{2})z}{\tau}\right)=\frac{\tau}{(m+\frac{1}{2})z}-\frac{(m+\frac{1}{2})z}{3\tau}+O(z^3).$$

Multiplying:

$$f_m(z)=\left(\frac{1}{(m+\frac{1}{2})z}-\frac{(m+\frac{1}{2})z}{3}+O(z^3)\right)\left(\frac{\tau}{(m+\frac{1}{2})z}-\frac{(m+\frac{1}{2})z}{3\tau}+O(z^3)\right).$$

Expanding,

$$f_m(z)=\frac{\tau}{(m+\frac{1}{2})^2z^2}-\frac{\tau}{3}-\frac{1}{3\tau}+O(z^2).$$

Since $g_m(z)=\frac{f_m(z)}{z}$, we extract the coefficient of $\frac{1}{z}$:

$$\operatorname{Res}_{z=0}g_m(z)=-\frac{1}{3}(\tau+\tau^{-1}).$$

e) Let Γ be the polygonal contour in $\mathbf{C}$ joining in counterclockwise order the vertices 1, τ, -1, $-\tau$ and 1 again. Prove that the functions g_m are uniformly bounded on Γ for all m, and prove that

$$\lim_{m\to+\infty}\int_\Gamma g_m(z)dz=\int_1^\tau\frac{dz}{z}-\int_\tau^{-1}\frac{dz}{z}+\int_{-1}^{-\tau}\frac{dz}{z}-\int_{-\tau}^{1}\frac{dz}{z}.$$

(Hint: compute the limit of $g_m(z)$ for z in Γ outside of the vertices.) Deduce the value, as a function of τ, of

$$\lim_{m\to+\infty}\exp\Big(3\int_\Gamma g_m(z)dz\Big).$$

Solution:
Since g_m only has poles at points on the real axis, each g_m is bounded on Γ. Moreover

$$\begin{aligned}\left|\operatorname{cotan}\left(\left(m+\frac{1}{2}\right)z\right)\right| &\leq 1+\frac{2}{|e^{-2i\left(m+\frac{1}{2}\right)z}-1|}\\ &\leq 1+\frac{2}{|e^{2i\left(m+\frac{1}{2}\right)\mathrm{Im}(z)}-1|}\\ &\leq 1+\frac{2}{|e^{2i\left(m+\frac{1}{2}\right)\mathrm{Im}(-\tau)}-1|}\end{aligned}$$

The expression on the right converges as $m \to \infty$, so that $\operatorname{cotan}((m+\frac{1}{2})z)$ is uniformly bounded for all $z \in \Gamma$.
An analogous argument shows that $\operatorname{cotan}((m+\frac{1}{2})z/\tau)$ is uniformly bounded. Since $1/z$ is also bounded on Γ, we obtain that g_m is uniformly bounded on Γ for all m.
Next, we compute $\lim_{m\to+\infty}\int_\Gamma g_m(z)dz$. Note that $\exp((m+\frac{1}{2})z) \to 0$ as $m \to \infty$ if and only if $\mathrm{Re}(z) < 0$. Thus $\operatorname{cotan}((m+\frac{1}{2})z) = i\left(\frac{2}{1-e^{-2i(m+\frac{1}{2})z}}-1\right) \to i$, as $m \to \infty$, if and only if $\mathrm{Im}(z) < 0$.
On the other hand, $\operatorname{cotan}((m+\frac{1}{2})z) \to -i$ if and only if $\mathrm{Im}(z) > 0$. Hence $\operatorname{cotan}((m+\frac{1}{2})z/\tau) \to i$, as $m \to \infty$, if and only if $\mathrm{Im}(z/\tau) < 0$, which holds if and only if $\mathrm{Im}(z\overline{\tau}) < 0$. Note that

$$\mathrm{Im}(z\overline{\tau}) = \mathrm{Im}(z)\mathrm{Re}(\overline{\tau}) - \mathrm{Re}(z)\mathrm{Im}(\tau),$$

which is greater than 0 if and only if

$$\mathrm{Im}(z)\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} > \mathrm{Re}(z)$$

From this expression, it is clear that for z above the straight line in the complex plane connecting τ to $-\tau$, we have that $\mathrm{Im}(z\overline{\tau}) > 0$. Hence $\operatorname{cotan}((m+\frac{1}{2})z)$ converges to i, as $m \to \infty$, if $z \in (-\tau, 1) \cup (1, \tau)$, and $\operatorname{cotan}((m+\frac{1}{2})z)$ converges to $-i$, as $m \to \infty$, if $z \in (\tau, -1) \cup (-1, \tau)$.
Thus we conclude that $f_m(z)$ converges to 1, as $m \to \infty$, on the line segments $1, \tau$ and $-1, -\tau$, and $f_m(z)$ converges to -1, as $m \to \infty$, on the line segments $\tau, -1$ and $-\tau, 1$.
By the dominated convergence theorem we have

$$\begin{aligned}\lim_{m\to\infty}\int_\Gamma f_m(z)\frac{dz}{z} &= \int_\Gamma \lim_{m\to\infty} f_m(z)\frac{dz}{z}\\ &= \left(\int_1^\tau - \int_\tau^{-1} + \int_{-1}^{-\tau} - \int_{-\tau}^1\right)\frac{dz}{z}\\ &= 2\left(\int_1^\tau - \int_{-\tau}^1\right)\frac{dz}{z}\\ &= 2(\log(\tau) + \log(-\tau))\\ &= 4\left(\log(\tau) - \frac{\pi i}{2}\right) = 4\log\left(\frac{\tau}{i}\right).\end{aligned}$$

Hence

$$\lim_{m\to\infty} \exp\left(3\int_\Gamma g_m(z)\right)dz = \exp\left(\log(\tau/i)\right)^{12}$$
$$= \tau^{12}$$

f) Prove that for all m, we have

$$\int_\Gamma g_m(z)dz = -\frac{2i\pi}{3}(\tau+\tau^{-1}) + 8\sum_{k=1}^{\lfloor\frac{m+1/2}{\pi}\rfloor}\frac{1}{k}\Big(\frac{1}{e(-k\tau)-1} - \frac{1}{e(k/\tau)-1}\Big).$$

Solution:
By the residue theorem we have

$$\int_\Gamma g_m(z)\frac{dz}{z} = -2\pi i\frac{\tau+\tau^{-1}}{3} + 2\pi i\sum_{k=1}^{\lfloor\frac{m+1/2}{\pi}\rfloor}\frac{2}{\pi}(\text{cotan}(\pi k\tau)+\text{cotan}(\pi k/\tau)),$$

which is equivalent to

$$\frac{2\pi i}{3}(\tau+\tau^{-1}) + \int_\Gamma g_m(z)\frac{dz}{z} = 4i\sum_{k=1}^{\lfloor\frac{m+1/2}{\pi}\rfloor}\frac{1}{k}(\text{cotan}(\pi k\tau)+\text{cotan}(\pi k/\tau))$$
$$= 8\sum_{k=1}^{\lfloor\frac{m+1/2}{\pi}\rfloor}\frac{1}{k}\left(\frac{1}{e(-k\tau)-1} - \frac{1}{e(k/\tau)-1}\right).$$

g) Deduce that

$$\lim_{m\to+\infty}\exp\Big(3\int_\Gamma g_m(z)dz\Big) = \frac{\Delta(-1/\tau)}{\Delta(\tau)},$$

and conclude that $\Delta \in M_{12}^0$. (This proof is due to Siegel.)
Solution:
We are grateful to Cajetan Tulej for suggesting this elegant solution.
We firstly obtain from part f) that

$$\lim_{m\to\infty}\exp\left(3\int_\Gamma g_m(z)dz\right) = e(-(\tau+\tau^{-1}))\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{e(-k\tau)-1} - \frac{1}{e(k/\tau)-1}\right)\right)^{24}.$$

Now observe that for all $k \geqslant 1$

$$|e(-k\tau)| = e^{2\pi k \text{Im}\tau} > 1,$$
$$|e(k/\tau)| = e^{2\pi k\text{Im}\tau/|\tau|^2} > 1$$

hold, so that using the geometric series expansion (after rewriting the terms), we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{e(-k\tau)-1}-\frac{1}{e(k/\tau)-1}\right) &= \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{e(k\tau)}{1-e(k\tau)}-\frac{e(-k/\tau)}{1-e(-k/\tau)}\right)\\ &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k}\left(e(k\tau)e(nk\tau)-e(-k/\tau)e(-kn/\tau)\right)\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left(e(nk\tau)-e(-kn/\tau)\right).\end{aligned}$$

Recalling that $\log(z-1) = -\sum_{k=1}^{\infty}\frac{z^k}{k}$ for $|z|<1$, we rewrite this further as

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left(e(nk\tau)-e(-kn/\tau)\right) = \sum_{n=1}^{\infty} -\log(e(n\tau)-1)+\log(e(-n/\tau)-1).$$

Inserting this into our initial equality, we find that

$$\begin{aligned}\lim_{m\to\infty}\exp\left(3\int_{\Gamma} g_m(z)dz\right) &= e(-(\tau+\tau^{-1}))\exp\left(\sum_{n=1}^{\infty} -\log(e(n\tau)-1)+\log(e(-n/\tau)-1)\right)^{24}\\ &= e(-(\tau+\tau^{-1}))\prod_{n\geqslant 1}\left(\frac{1-e(-n/\tau)}{1-e(n\tau)}\right)^{24}\\ &= \frac{e(-1/\tau)}{e(\tau)}\cdot\frac{\prod_{n\geqslant 1}(1-e(-n/\tau))^{24}}{\prod_{n\geqslant 1}(1-e(n\tau))^{24}}\\ &= \frac{\Delta(-1/\tau)}{\Delta(\tau)}\end{aligned}$$

as was to be shown. From part e) it then follows that

$$\frac{\Delta(-1/\tau)}{\Delta(\tau)} = \tau^{12} \quad\rightsquigarrow\quad \Delta\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\tau\right) = \Delta(-1/\tau) = \tau^{12}\Delta(\tau).$$

On the other hand, we clearly also have by periodicity of e that

$$\Delta\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\tau\right) = 1^{12}\Delta(\tau).$$

As these two matrices generate $\mathrm{PSL}_2(\mathbb{Z})$ and $\tau\in\mathbb{H}$ was arbitrary, we conclude that Δ defines a modular form of weight 12. (The above relation $\tau^{12} = \frac{\Delta(-1/\tau)}{\Delta(\tau)}$ also implies that it is not the zero function.)

It remains to be shown that Δ is a cusp form. Transferring Δ to $\widetilde{\Delta}:\mathbf{D}^*\to\mathbb{C}$, given by

$$\widetilde{\Delta}(w) = w\prod_{n\geqslant 1}(1-w^n)^{24},$$

we see that it has continuation to 0, namely $\widetilde{\Delta}(0)=0$, which proves $\Delta\in M_{12}^0$.