# Solutions: Exercise Sheet 6

**1.** Let $k \geq 4$ be an even integer and let $f \in \mathcal{H}_k$ be a Hecke eigenform (i.e., an element of the Hecke basis of $S_k(1)$). Let $\lambda_f(n)$ denote the eigenvalue for the $n$-th Hecke operator. Let $\delta \geq 0$ be a real number such that $\lambda_f(n) = O(n^\delta)$ for $n \geq 1$.

a) For any prime number $p$, prove that the power series

$$\sum_{\nu \geq 0} \lambda(p^\nu) X^\nu$$

has radius of convergence $\geq p^{-\delta}$.

Solution:

Since $\lambda_f(p^\nu) = O(p^{\delta\nu})$, for all positive integers $\nu \geq 0$, there exists a positive constant $C \geq 1$ such that $|\lambda_f(p^\nu)| \leq C \cdot p^{\lambda\nu}$. Then we have the following inequality for the radius of convergence $r$

$$r = \frac{1}{\limsup\limits_{\lambda \to \infty} \sqrt[\nu]{|\lambda_f(p^\nu)|}} \geq \frac{1}{\limsup\limits_{\lambda \to \infty} \sqrt[\nu]{C p^{\delta\nu}}} \geq p^{-\delta}$$

b) Deduce that $|\lambda_f(n)| \leq d(n) n^\delta$ for all $n \geq 1$ (here $d$ is the divisor function).

Solution:

Recall from the lectures that for each $\nu \geq 0$ we can write

$$\lambda_f(p^\nu) = \sum_{j=0}^{\nu} \alpha_p^j \beta_p^{\nu - j},$$

where $\alpha_p = e^{i\vartheta_p}, \beta_p = e^{-i\vartheta_p}$, for some complex number $\vartheta_p$.

First we will show that $|\alpha_p| = |\beta_p^{-1}| = p^{-\delta}$. Assume $|\alpha_p| \neq p^{-\delta}$. Then we can assume without loss of generality that $\alpha_p$ is such that $|\alpha_p| < p^{-\delta}$, so that $|\beta_p| > p^\delta$. Taking $X = \alpha_p$, we can write

$$\begin{aligned}
\lambda_f(p^\nu) X^\nu &= \alpha_p^\nu \sum_{j=0}^{\nu} \alpha_p^j \beta_p^{\nu - j} \\
&= \sum_{j=0}^{\nu} \alpha_p^j \beta_p^{-j},
\end{aligned}$$

since $\alpha_p\beta_p = \alpha_p^\nu\beta_p^\nu = 1$. Further write

$$\sum_{j=0}^{\nu} \alpha_p^j \beta_p^{-j} = \sum_{j=0}^{\nu} \beta_p^{-2j} = \frac{1-\beta_p^{-(2\nu+2)}}{1-\beta_p^{-2}},$$

which converges to 1 as $\nu \to +\infty$ (since $|\beta_p| > p^\delta$). But then $\sum_{\nu\geq 0} \lambda_f(p^\nu)X^\nu$ diverges, which is a contradiction to this power series having radius of convergence equal to $p^{-\delta}$ by part a). Hence $|\alpha_p| = |\beta_p^{-1}| = p^{-\delta}$.

Then

$$|\lambda_f(p^\nu)| = \left|\sum_{j=0}^{\nu} \alpha_p^j \beta_p^{\nu-j}\right| \leq \sum_{j=0}^{\nu} |\alpha_p|^j |\beta_p|^{\nu-j} = \sum_{j=0}^{\nu} p^{-\delta j} p^{\delta\nu-\delta j} = p^{\delta\nu}(\nu+1) = p^{\delta\nu} d(p^\nu).$$

By multiplicativity we have $|\lambda_f(n)| \leq n^\delta d(n)$ for all $n \geq 1$.

c) Deduce that the following statements are equivalent:

(i) For all cusp forms $g \in S_k(1)$, the Fourier coefficients $a_g(n)$ of $g$ satisfy the estimate $a_g(n) = O(n^{(k-1)/2+\varepsilon})$ for all $\varepsilon > 0$.

(ii) For all $f \in \mathcal{H}_k$ and for all $n \geq 1$, we have $|\lambda_f(n)| \leq d(n)n^{(k-1)/2}$.

Solution:

$''(i) \implies (ii)''$ Let $f \in \mathcal{H}_k$ be a Hecke eigenform. Then $f \in S_k(1)$ as well, so by part $(i)$ we have for all $\varepsilon > 0$ that

$$|\lambda_f(n)| = |a_f(n)| = O(n^{(k-1)/2+\varepsilon}),$$

so by part b) it follows that $|\lambda_f(n)| \leq d(n)n^{(k-1)/2}$.

$''(ii) \implies (i)''$ Let $g = \sum_m a_g(m)q^m \in S_k(1)$ be a cusp form. From the lectures we know that there exist Hecke eigenforms $\{f_i\}_i \in \mathcal{H}_k$ such that

$$g = \sum_i b_i f_i$$

Then writing for each $i$ the Fourier expansion $f_i = \sum_n a_i(n)q^n$, note that

$$a_g(m) = \sum_i b_i a_i(m)$$

Moreover, since each $a_i(m)$ has a corresponding Hecke eigenvalue $\lambda_i(m)$, we obtain by part (ii)

$$|a_g(m)| \leq \sum_i |b_i||\lambda_i(m)| \leq \sum_i |b_i| d(m) m^{(k-1)/2} = O(m^{(k-1)/2+\varepsilon}),$$

since $d(m) = o(m^\varepsilon)$, for each $\varepsilon > 0$.

**2.** Let $q \geq 1$ be an integer.

a) Let $\chi$ be a Dirichlet character modulo $q$. For any integer $d \geq 1$, show that the function $\widetilde{\chi}$ defined on integers by

$$\widetilde{\chi}(n) = \begin{cases} \chi(n) & \text{if } (n, qd) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbf{Z}$ is a Dirichlet character modulo $qd$. It is said to be *induced* by $\chi$. [1]

Solution:

Let $a, b \in \mathbb{Z}$ be arbitrary. First we will show that $\tilde{\chi}$ is multiplicative. By definition, we have that $\tilde{\chi}(ab)$ is non-zero if and only if $ab$ is coprime to $qd$, which holds if and only if both $a$ and $b$ are coprime to $qd$. If that is the case, then we have

$$\tilde{\chi}(ab) = \chi(ab) = \chi(a)\chi(b) = \tilde{\chi}(a)\tilde{\chi}(b).$$

Hence $\tilde{\chi}$ is multiplicative. We are left to show that $\tilde{\chi}$ has period $qd$. Let $a, n \in \mathbf{Z}$ be arbitrary integers. Since $(a + nqd, qd) = 1$ if and only if $(a, qd) = 1$, together with the fact that $\chi(a + nqd) = \chi(a)$, we have that $\tilde{\chi}(a + nqd) = \tilde{\chi}(a)$.

b) Compute $L(s, \widetilde{\chi})$ in terms of $L(s, \chi)$.

Solution:

By definition, we have

$$\begin{aligned} L(s, \tilde{\chi}) = \sum_{n \geq 1} \frac{\tilde{\chi}(n)}{n^s} &= \sum_{\substack{n \geq 1 \\ (n, dq) = 1}} \frac{\chi(n)}{n^s} \\ &= \sum_{\substack{n \geq 1 \\ (n, d) = 1}} \frac{\chi(n)}{n^s} \end{aligned}$$

since $\chi$ is a Dirichlet character modulo $q$, we have $\chi(n) = 0$ if $(n, q) \neq 1$. The above can then be rewritten to

$$\begin{aligned} \sum_{\substack{n \geq 1 \\ (n, d) = 1}} \frac{\chi(n)}{n^s} = \sum_{n \geq 1} \frac{\chi(n) \mathbf{1}_{(n,d)=1}}{n^s} &= \prod_{p \nmid d} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \\ &= \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \prod_{p \mid d} \left(1 - \frac{\chi(p)}{p^s}\right) \\ &= L(s, \chi) \prod_{p \mid d} \left(1 - \frac{\chi(p)}{p^s}\right). \end{aligned}$$

A Dirichlet character modulo $q$ is said to be *primitive* if there does not exist a divisor $d \mid q$ with $d < q$ and a Dirichlet character $\eta$ modulo $d$ such that $\chi$ is induced by $\eta$.

[1] Warning: this is not the standard meaning of the word “induce” in representation theory.

c) Explain why the primitive characters modulo a *prime* $q$ are the non-trivial characters modulo $q$.

Solution:

Since $q$ is prime, the only divisor $d \mid q$ with $d < q$ is $d = 1$. The only Dirichlet character $\eta$ modulo 1 is the constant character, i.e. for all $n \in \mathbf{Z}$ we have $\eta(n) = 1$. This induces only one character modulo $q$, $\tilde{\eta}$, which is $\tilde{\eta}(n) = 1$ if $(n, q) = 1$, and $\tilde{\eta}(n) = 0$ otherwise (hence the trivial character modulo $q$).

d) Show that if $\chi$ is any Dirichlet character modulo $q$, there exists a unique divisor $d \mid q$ (possibly equal to $q$) and primitive Dirichlet character $\eta$ modulo $d$ such that $\chi$ is induced from $\eta$. (Hint: to prove uniqueness, show that if $\chi$ is induced from characters $\eta_1$ modulo $d_1$ and $\eta_2$ modulo $d_2$, then it is induced from a character modulo the greatest common divisor $\gcd(d_1, d_2)$.)

Solution:

Let $\chi$ be a Dirichlet character modulo $q$.

First we will show existence of such a character. If $\chi$ is primitive, we can choose $d = q$ and $\eta = \chi$. So we claim that $\chi$ is not primitive. Consider the set $D$ of all divisors $d \mid q$ for which $\chi$ is induced by some Dirichlet character $\eta$ modulo $d$. Note that this set is not empty, since $q \in D$.

Let $d' \in D$ be minimal such that for all $d \in D$ we have $d \geq d'$ and $\chi$ is induced by some Dirichlet character $\eta'$ modulo $d'$. We claim that $\eta'$ is primitive. If it was not primitive, then there exists a divisor $r' < d'$ of $d'$ and a Dirichlet character $\gamma$ modulo $r'$ such that for all $n \in \mathbf{Z}$ we have

$$\eta'(n) = \begin{cases} \gamma(n) & (n, d') = 1 \\ 0 & \text{otherwise} \end{cases}$$

But then

$$\chi(n) = \begin{cases} \eta'(n) & (n, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Combining this with the formula for $\eta'$ we stated above, we obtain that

$$\chi(n) \begin{cases} \gamma(n) & (n, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence the character $\chi$ is induced by $\gamma$, which is a character modulo $r' < d'$, which is a contradiction to the minimality of $d'$. Hence $\eta'$ is primitive.

Next we will show uniqueness. Suppose that $\chi$ is induced from primitive characters $\eta_1$ modulo $d_1$ and $\eta_2$ modulo $d_2$. Then for all $(n, q) = 1$ we have $\chi(n) = \eta_1(n) = \eta_2(n)$. Let $d := \gcd(d_1, d_2)$. Composing the character $\eta_i$ with the reduction modulo $d$ homomorphism, we get that $\eta_i$ is induced by a Dirichlet character modulo $d$. Since both $\eta_1$ and $\eta_2$ are primitive, we get that $d = d_1 = d_2$.

Hence we are left to show that for every $n \in \mathbf{Z}$ we have $\eta_1(n) = \eta_2(n)$. Let $n \in \mathbf{Z}$ be arbitrary such that $(n, d) = 1$, and let $m \in \mathbf{Z}$ be an integer such that $(n + md, q) = 1$. Then

$$\eta_1(n) = \eta_1(n + md) = \chi(n + md) = \eta_2(n + md) = \eta_2(n)$$

e) Let $\varphi^*(q)$ denote the number of primitive characters modulo $q$. Show that

$$\varphi(q) = \sum_{d|q} \varphi^*(d)$$

and

$$\varphi^*(q) = \sum_{d|q} \mu(d)\varphi(q/d).$$

Solution:

By part d) we know that $\varphi(q)$ is given by the number of Dirichlet characters modulo $q$, which in turn, is equal to the number of Dirichlet characters modulo $q$, which are induced by a primitive Dirichlet character modulo $d$, for all divisors $d \mid q$. In other words,

$$\varphi(q) = \sum_{d|q} \varphi^*(d). \tag{1}$$

The second equation in the statement of the exercise is given by taking Möbius inversion of (1).

f) Determine the integers for which $\varphi^*(q) = 0$.

Solution:

By part e) together with the fact that both $\mu$ and $\varphi$ are multiplicative, we obtain that $\varphi^*$ is multiplicative:

$$\varphi^*(q) = \prod_{p|q} \varphi^*(p^{v_p(q)}).$$

For a prime number $p$ we have (using the formula from part e)) $\varphi^*(p) = \mu(1)\varphi(p) + \mu(p)\varphi(1) = p - 1 - 1 = p - 2$. Moreover

$$\begin{aligned}
\varphi^*(p^v) &= \sum_{n=0}^{v} \mu(p^n)\varphi(p^{v-n}) \\
&= \mu(1)\varphi(p^v) + \mu(p)\varphi(p^{v-1}) \\
&= \left(1 - \frac{1}{p}\right) p^v - \left(1 - \frac{1}{p}\right) p^{v-1} = p^{v-2}(p-1)^2
\end{aligned}$$

Hence we obtain the formula

$$\begin{aligned}
\varphi^*(q) &= \left( \prod_{v_p(q)=1} (p-2) \right) \left( \prod_{v_p(q)=2} p^{v_p(q)-2}(p-1)^2 \right) \\
&= q \prod_{p||q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2
\end{aligned}$$

where the products are taken over all primes $p$ with the stated conditions underneath the respective product symbols. Hence $\varphi^*(q) = 0$ if and only if $v_2(q) = 1$.

g) Let $\chi$ be a primitive Dirichlet character modulo $q$. Define

$$\tau(\chi) = \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{x}{q}\Big).$$

Show that $\overline{\chi}$ is also primitive, and prove that for any integer $a$ coprime to $q$, the formula

$$\tau(\overline{\chi})\chi(a) = \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \overline{\chi}(x) e\Big(\frac{ax}{q}\Big)$$

holds.

Solution:

Let $\chi$ be a primitive Dirichlet character modulo $q$. Then we have that $\chi$ is induced by a Dirichlet character $\eta$ modulo $d \mid q$ if and only if $\overline{\chi}$ is induced by a Dirichlet character $\overline{\eta}$ modulo $d \mid q$. From this equivalence it follows that $\overline{\chi}$ is primitive as well.

Similar to Exercise 2b) from Exercise sheet 4, we obtain

$$\begin{aligned}
\tau(\overline{\chi})\chi(a) &= \sum_{z \in \mathbf{Z}/q\mathbf{Z}} \chi(a)\overline{\chi}(x) e\left(\frac{x}{q}\right) \\
&= \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \overline{\chi}(xa^{-1}) e\left(\frac{x}{q}\right) \\
&= \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \overline{\chi}(x) e\left(\frac{ax}{q}\right)
\end{aligned}$$

h) Let $\chi$ be a primitive Dirichlet character modulo $q$. Let $a$ be an integer such that $d = (a, q) \geq 2$. Let $q'$ be defined by $q = dq'$.

Prove that for any $x'$ modulo $q'$, we have

$$\sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \bmod q'}} \chi(x) = 0.$$

(Hint: here, you have to use the fact that $\chi$ is primitive.)

Deduce that

$$\sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{ax}{q}\Big) = 0.$$

Solution:

Let $\chi, q$ and $q'$ be as above. We claim that there exists $y \in \mathbf{Z}$ such that $y \equiv 1 \pmod{q'}$ with $\chi(y) \neq 1$. If not, then we would have for every $y \in \mathbf{Z}$ with $y \equiv 1 \pmod{q'}$ that $\chi(y) = 1$. This in turn means that $\chi$ is induced by a character modulo $q'$. More precisely, let $n \in \mathbf{Z}$ be such that $(n, q) = 1$. Let $m \in \mathbf{Z}$ be arbitrary. Then there exists an inverse $n^{-1} \in \mathbf{Z}/q\mathbf{Z}$ such that $nn^{-1} = 1 + ql$, for some $l \in \mathbf{Z}$. Hence

$$\chi(n + q') = \chi((1 + q'n^{-1})n) = \chi(1 + q'n^{-1})\chi(n) = \chi(n).$$

Note that since $\chi(n) \neq 0$, we also have $\chi(n+q') \neq 0$, so $(n+q', q) = 1$.
By induction we obtain that $\chi(n+mq') = \chi(n)$, so $\chi$ indeed factors through $\mathbf{Z}/q'\mathbf{Z}$, which is a contradiction to $\chi$ being primitive. Hence we obtain our claim, and we can choose $y \in \mathbf{Z}$ such that $y \equiv 1 \pmod{q'}$ with $\chi(y) \neq 1$. Then

$$\sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \bmod q'}} \chi(x) = \sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \bmod q'}} \chi(yx) = \chi(y) \sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \bmod q'}} \chi(x),$$

and since $\chi(y) \neq 1$, we must have $\displaystyle\sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \bmod q'}} \chi(x) = 0$.

Next we will consider the second sum in the exercise. Write

$$\sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\left(\frac{ax}{q}\right) = \sum_{x' \in \mathbf{Z}/q'\mathbf{Z}} \sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \pmod{q'}}} \chi(x) e\left(\frac{ax}{q}\right)$$

Write $a = a'd$ and note that

$$e\left(\frac{ax}{q}\right) = e\left(\frac{a'x'}{q'}\right) e(a'n) = e\left(\frac{a'x'}{q'}\right),$$

for $n \in \mathbf{Z}$ such that $x = x' + nq'$. Hence

$$\sum_{x' \in \mathbf{Z}/q'\mathbf{Z}} \sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \pmod{q'}}} \chi(x) e\left(\frac{ax}{q}\right) = \sum_{x' \in \mathbf{Z}/q'\mathbf{Z}} e\left(\frac{a'x'}{q'}\right) \sum_{\substack{x \in \mathbf{Z}/q\mathbf{Z} \\ x \equiv x' \pmod{q'}}} \chi(x) = 0.$$

i) Conclude that if $\chi$ is a primitive Dirichlet character modulo $q$, then $\tau(\chi)$ and $\tau(\overline{\chi})$ are non-zero, and the formula

$$\chi(a) = \frac{1}{\tau(\overline{\chi})} \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\Big(\frac{ax}{q}\Big)$$

holds for all integers $a \in \mathbf{Z}$. *(This fact is very often the key property of primitive characters.)*

Solution:

Let $\chi$ be a primitive Dirichlet character modulo $q$. Then $\overline{\chi}$ is primitive as well. Moreover, $\chi$ is not the trivial character modulo $q$, so by Exercise 2 of Exercise Sheet 4, we have that $|\tau(\chi)| = |\tau(\overline{\chi})| = \sqrt{q} \neq 0$. By part g) and h) we have for $(a, q) = 1$ that

$$\frac{1}{\tau(\overline{\chi})} \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\left(\frac{ax}{q}\right) = \chi(a),$$

while for $(a, q) > 1$

$$\frac{1}{\tau(\overline{\chi})} \sum_{x \in \mathbf{Z}/q\mathbf{Z}} \chi(x) e\left(\frac{ax}{q}\right) = 0 = \chi(a).$$

Hence we obtain the formula in the exercise.