Solutions: Exercise Sheet 7

1. Let α and β be complex numbers. Define $(a_n)_{n\geq 0}$ by the power series expansion

$$\frac{1}{(1-\alpha X)(1-\beta X)} = \sum_{n\geq 0} a_n X^n.$$

a) Prove that

$$\sum_{n\geq 0} a_n^2 X^n = \frac{1-(\alpha\beta X)^2}{(1-\alpha^2 X)(1-\alpha\beta X)^2(1-\beta^2 X)}.$$

Solution:

We start by proving the following formula for the n'th coefficient a_n:

$$a_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \tag{1}$$

Writing out geometric sum expansions we obtain

$$\begin{aligned}\frac{1}{1-\alpha X}\frac{1}{1-\beta X} &= \left(\sum_{i\geq 0}(\alpha X)^i\right)\left(\sum_{j\geq 0}(\beta X)^j\right)\\ &= \sum_{n\geq 0} X^n \sum_{k=0}^{n} \alpha^k \beta^{n-k}\\ &= \sum_{n\geq 0} X^n \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\end{aligned}$$

Hence we obtain the formula (1), so that

$$a_n^2 = \frac{\alpha^{2n+2}-2(\alpha\beta)^{n+1}+\beta^{2n+2}}{(\alpha-\beta)^2}$$

Using this we can write

$$\begin{aligned}
\sum_{n\geq 0} a_n^2 &= \frac{1}{(\alpha-\beta)^2}\left(\sum_{n\geq 0}\alpha^{2n+2}X^n - \sum_{n\geq 0} 2(\alpha\beta)^{n+1}X^n + \sum_{n\geq 0}\beta^{2n+2}X^n\right)\\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{\alpha^2}{1-\alpha^2X} - \frac{2\alpha\beta}{1-\alpha\beta X} + \frac{\beta^2}{1-\beta^2X}\right)\\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{\alpha^2-2\alpha^2\beta^2X+\beta^2+\alpha^3\beta X-2\alpha\beta+\alpha\beta^3X}{(1-\alpha^2X)(1-\alpha\beta X)(1-\beta^2X)}\right)\\
&= \frac{1}{(\alpha-\beta)^2}\left(\frac{(\alpha-\beta)^2+\alpha\beta X(\alpha-\beta)^2}{(1-\alpha^2X)(1-\alpha\beta X)(1-\beta^2X)}\right)\\
&= \frac{1-(\alpha\beta X)^2}{(1-\alpha^2X)(1-\alpha\beta X)^2(1-\beta^2X)}
\end{aligned}$$

b) Prove that

$$\sum_{n\geq 0} a_{2n}X^n = \frac{1-(\alpha\beta X)^2}{(1-\alpha^2X)(1-\alpha\beta X)(1-\beta^2X)}.$$

Solution:
Using the formula (1) we computed in part a) we have

$$a_{2n} = \frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}$$

Then proceeding as in part a)

$$\begin{aligned}
\sum_{n\geq 0} a_{2n}X^n &= \frac{1}{\alpha-\beta}\left(\sum_{n\geq 0}\alpha^{2n+1}X^n - \sum_{n\geq 0}\beta^{2n+1}X^n\right)\\
&= \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha^2X} - \frac{\beta}{1-\beta^2X}\right)\\
&= \frac{1}{\alpha-\beta}\frac{\alpha-\beta+\alpha\beta X(\alpha-\beta)}{(1-\alpha^2X)(1-\beta^2X)}\\
&= \frac{1-(\alpha\beta X)^2}{(1-\alpha^2X)(1-\alpha\beta X)(1-\beta^2X)}
\end{aligned}$$

c) Let $d(n)$ denote the divisor function (number of $d \mid n$ with $d \geq 1$). Prove that

$$\sum_{n\geq 1}\frac{d(n^2)}{n^s} = \frac{\zeta(s)^3}{\zeta(2s)}$$

for $\mathrm{Re}(s) > 1$.
Solution:
Let s be a complex number such that $\mathrm{Re}(s) > 1$. Since the divisor function is multiplicative, in the sense that $d(ab) = d(a)d(b)$ for a, b coprime, we can write

$$\sum_{n\geq 1}\frac{d(n^2)}{n^s} = \prod_{p\text{ prime}}\sum_{k\geq 0}\frac{d(p^{2k})}{p^{ks}} = \prod_{p}\sum_{k\geq 0}\frac{2k+1}{p^{ks}}$$

Writing $X := p^{-s}$, we can rewrite the geometric sums

$$\begin{aligned}
\sum_{k\geq 0}(2k+1)X^k &= 2\sum_{k\geq 0} kX^k + \sum_{k\geq 0} X^k \\
&= 2X \cdot \frac{d}{dX}\left(\sum_{k\geq 0} X^k\right) + \sum_{k\geq 0} X^k \\
&= 2X \cdot \frac{d}{dX}\left(\frac{1}{1-X}\right) + \frac{1}{1-X} \\
&= 2X\frac{1}{(1-X)^2} + \frac{1}{1-X} \\
&= \frac{1+X}{(1-X)^2}
\end{aligned}$$

Hence

$$\prod_p \sum_{k\geq 0} \frac{2k+1}{p^{ks}} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^2} = \prod_p \frac{1-p^{-2s}}{(1-p^{-2})^3} = \frac{\zeta(s)^3}{\zeta(2s)}$$

2. Let $q \geq 2$ be an integer and let χ be a primitive Dirichlet character modulo q. Assume that $\chi(-1) = 1$.

Let

$$\Lambda(s,\chi) = q^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi).$$

a) Show that

$$\Lambda(s,\chi) = \epsilon\Lambda(1-s,\bar{\chi})$$

where ϵ is a complex number with modulus 1.

Solution:

We have seen in the lectures that

$$\begin{aligned}
\Lambda(s,\chi) &= q^{\frac{s}{2}} \int_0^\infty \left(\sum_{n\geq 0} \chi(n)e^{-\pi n^2 t}\right) t^{\frac{s}{2}} \frac{dt}{t} \\
&= \frac{q^{\frac{s}{2}}}{2} \int_0^\infty \vartheta(it,\chi)t^{\frac{s}{2}}\frac{dt}{t} \\
&= \frac{q^{\frac{s}{2}}}{2} \left(\int_0^{\frac{1}{q}} \vartheta(it,\chi)t^{\frac{s}{2}}\frac{dt}{t} + \int_{\frac{1}{q}}^\infty \vartheta(it,\chi)t^{\frac{s}{2}}\frac{dt}{t}\right)
\end{aligned}$$

Using the transformation formula from the lectures $\vartheta(it,\chi) = \frac{\tau(\chi)}{q\sqrt{t}}\vartheta\left(\frac{i}{q^2t},\overline{\chi}\right)$ together with a change of variables from t to $1/(q^2t)$, we have

$$\begin{aligned}\int_0^{\frac{1}{q}} \vartheta(it,\chi)t^{\frac{s}{2}}\frac{dt}{t} &= \frac{\tau(\chi)}{q}\int_0^{\frac{1}{q}} \vartheta\left(\frac{i}{q^2t},\overline{\chi}\right)t^{\frac{-1+s}{2}-1}dt \\ &= \frac{\tau(\chi)}{q}\int_{\frac{1}{q}}^{\infty} \vartheta(it,\overline{\chi})\left(\frac{1}{q^2t}\right)^{\frac{-1+s}{2}-1}\cdot\frac{-1}{q^2t^2}dt \\ &= \frac{-\tau(\chi)}{q^s}\int_{\frac{1}{q}}^{\infty} \vartheta(it,\overline{\chi})t^{\frac{-s+1}{2}-1}dt\end{aligned}$$

Putting it all together we obtain

$$\begin{aligned}\Lambda(s,\chi) &= \frac{q^{\frac{s}{2}}}{2}\left(\frac{-\tau(\chi)}{q^s}\int_{\frac{1}{q}}^{\infty}\vartheta(it,\overline{\chi})t^{\frac{-s+1}{2}-1}dt + \int_{\frac{1}{q}}^{\infty}\vartheta(it,\chi)t^{\frac{s}{2}-1}dt\right) \\ &= \frac{1}{2}\int_{\frac{1}{q}}^{\infty}\left(\frac{-\tau(\chi)}{\sqrt{q}}q^{\frac{1-s}{2}}\vartheta(it,\overline{\chi})t^{\frac{-s+1}{2}} + q^{\frac{s}{2}}\vartheta(it,\chi)t^{\frac{s}{2}}\right)\frac{dt}{t}\end{aligned}$$

On the other hand, using the fact that $\tau(\chi)\tau(\overline{\chi}) = q$, we have

$$\begin{aligned}\frac{\tau(\chi)}{\sqrt{q}}\Lambda(1-s,\overline{\chi}) &= \frac{1}{2}\int_{\frac{1}{q}}^{\infty}\left(\frac{-\tau(\chi)\tau(\overline{\chi})}{q^{\frac{1}{2}}q^{\frac{1-s}{2}}}\vartheta(it,\chi)t^{\frac{1(1-s)+1}{2}} + q^{\frac{1-s}{2}}\frac{\tau(\chi)}{\sqrt{q}}\vartheta(it,\overline{\chi})t^{\frac{1-s}{2}}\right)\frac{dt}{t} \\ &= \frac{-1}{2}\int_{\frac{1}{q}}^{\infty}\left(q^{\frac{s}{2}}\vartheta(it,\chi)t^{\frac{s}{2}} - \frac{\tau(\chi)}{\sqrt{q}}q^{\frac{1-s}{2}}\vartheta(it,\overline{\chi})t^{\frac{1-s}{2}}\right)\frac{dt}{t}\end{aligned}$$

So that

$$\Lambda(s,\chi) = \frac{-\tau(\chi)}{\sqrt{q}}\Lambda(1-s,\overline{\chi})$$

b) Let $X \geq 1$ be a real number. Show that the integral

$$I(X,\chi) = \frac{1}{2i\pi}\int_{(3)}\Lambda(s+1/2,\chi)\hat{\phi}(s)X^s\frac{ds}{s}$$

exists, where the integral is over the line with real part 3. [Hint: you may use the complex Stirling formula for the Gamma function, in the form of the estimate

$$\Gamma(\sigma+it) = \sqrt{2\pi}(it)^{\sigma-1/2}e^{-\pi|t|/2}\left(\frac{|t|}{e}\right)^{it}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

valid for $\sigma \in \mathbf{R}$ fixed and $t > 0$.]

Solution:

Writing $s := 3+it$, we have

$$\begin{aligned}I(X,\chi) &= \frac{1}{2i\pi}\int_{(3)}\Lambda(s+1/2,\chi)\hat{\varphi}(s)X^s\frac{ds}{s} \\ &= \frac{1}{2i\pi}\int_{-\infty}^{+\infty}\Lambda(3+1/2+it,\chi)\hat{\varphi}(3+it)X^{3+it}\frac{dt}{3+it}\end{aligned}$$

Using the complex Stirling formula we obtain

$$|\Gamma(3+1/2+it)| = \sqrt{2\pi} t^{1+\frac{1}{4}} e^{\frac{-\pi|t|}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right)$$

so that together with the estimate $|L(3+1/2+,\chi)| = \left|\sum_{n\geq 1} \frac{\chi(n)}{n^{3+\frac{1}{2}+it}}\right| \leq \sum_{n\geq 1} \frac{1}{n^3} \ll 1$ we have

$$|\Lambda(3+1/2+it)| \leq \left(\frac{q}{\pi}\right)^{(3+1/2)/2} \sqrt{2\pi} t^{1+\frac{1}{4}} e^{\frac{-\pi|t|}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right)$$

Since Schwartz functions decrease rapidly, in the sense that $|\hat{\varphi}(x)| \ll x^{-k}$, for any $k \in \mathbf{N}$, we have

$$\begin{aligned} |I(X,\chi)| &\leq \frac{X^3}{2\pi} \int_{-\infty}^{+\infty} |\Lambda(3+1/2+it,\chi)||\hat{\varphi}(3+it)| \frac{1}{\sqrt{9+t^2}} dt \\ &\ll X^3 \int_{-\infty}^{+\infty} t^{1+\frac{1}{4}} e^{\frac{-\pi|t|}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right) 3^{-k} \frac{1}{\sqrt{9+t^2}} dt \end{aligned}$$

and the integral above converges.

c) Prove that

$$I(X,\chi) + \epsilon I(X^{-1},\bar{\chi}) = \Lambda(1/2,\chi).$$

[Hint: you may use the fact that $L(s,\chi)$ has polynomial growth in any region $\mathrm{Re}(s) \geq A$, where $A \in \mathbf{R}$ may be negative (this was proved in Number Theory I for $A > 0$).]

Solution:

We will ßhiftthe line of integration to the line $\mathrm{Re}(s) = 3$. More precisely, since $L(s,\chi)$ has polynomial growth, together with Cauchy's theorem, we have

$$\begin{aligned} I(X,\chi) &= \mathrm{Res}_{s=0} \frac{\Lambda(s+1)\hat{\varphi}(s)X^s s^{-1}}{s} + \frac{1}{2\pi i} \int_{(-3)} \Lambda(s+1/2,\chi)\hat{\varphi}(s)X^s s^{-1} ds \\ &= \Lambda(1/2,\chi) + \frac{1}{2\pi i} \int_{(-3)} \Lambda(s+1/2,\chi)\hat{\varphi}(s)X^s s^{-1} ds \end{aligned}$$

Changing variables from s to $-s$ we obtain

$$\Lambda(1/2,\chi) = I(X,\chi) + \frac{1}{2\pi i} \int_{(3)} \Lambda(-s+1/2,\chi)\hat{\varphi}(-s)X^{-s}s^{-1} ds$$

Using the functional equation from part a) we have

$$\begin{aligned} I(1/2,\chi) &= I(X,\chi) + \frac{1}{2\pi i} \int_{(3)} \varepsilon\Lambda(s+1/2,\overline{\chi})\hat{\varphi}(s)X^{-s}s^{-1} ds \\ &= I(X,\chi) + \varepsilon I(X^{-1},\overline{\chi}). \end{aligned}$$

d) Prove that

$$I(X,\chi) = q^{1/2}\pi^{-1/4}\Gamma(1/4)\sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}V\Big(\frac{n}{X\sqrt{q}}\Big)$$

where

$$V(y) = \frac{1}{2i\pi}\int_{(3)} y^{-s}\pi^{-1/4}\frac{\Gamma(s/2+1/4)}{\Gamma(s/2)}\frac{ds}{s}$$

Solution:

An explicit computation, together with expanding $L(s+1/2,\chi)$ gives

$$\begin{aligned}
I(X,\chi) &= \frac{1}{2i\pi}\int_{(3)}\Lambda(s+1/2,\chi)X^s s^{-1}ds \\
&= \frac{1}{2i\pi}\int_{(3)}\left(\frac{q}{\pi}\right)^{\frac{s+1/2}{2}}\Gamma\left(\frac{s+1/2}{2}\right)\left(\sum_{n\geq 1}\chi(n)n^{-(s+1/2)}\right)X^s s^{-1}ds \\
&= \left(\frac{q}{\pi}\right)^{\frac{1}{4}}\sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}\frac{1}{2i\pi}\int_{(3)}\left(\frac{q}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)n^{-s}X^s\frac{ds}{s} \\
&= q^{1/4}\pi^{-1/4}\Gamma(1/4)\sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}\left(\frac{1}{2i\pi}\int_{(3)}(q^{1/2}Xn^{-1})^s\pi^{-(\frac{s}{2}+\frac{1}{4})}\frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)}{\Gamma(1/4)\pi^{-1/4}}\frac{ds}{s}\right) \\
&= q^{1/4}\pi^{-1/4}\Gamma(1/4)\sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}V\left(\frac{n}{X\sqrt{q}}\right)
\end{aligned}$$

e) Deduce that

$$L(1/2,\chi) = \sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}V\Big(\frac{n}{X\sqrt{q}}\Big) + \epsilon\sum_{n\geq 1}\frac{\overline{\chi(n)}}{\sqrt{n}}V\Big(\frac{nX}{\sqrt{q}}\Big).$$

Solution:

Using part c) together with applying the formula from part d) to $I(X^{-1},\overline{\chi})$ we obtain

$$\begin{aligned}
\Lambda(1/2,\chi) &= I(X,\chi) + \epsilon I(X^{-1},\bar{\chi}) \\
&= q^{1/4}\pi^{-1/4}\Gamma(1/4)\left(\sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}V\left(\frac{n}{X\sqrt{q}}\right) + \epsilon\sum_{n\geq 1}\frac{\overline{\chi}(n)}{\sqrt{n}}V\left(\frac{nX}{\sqrt{q}}\right)\right) \\
&= \sum_{n\geq 1}\frac{\chi(n)}{\sqrt{n}}V\Big(\frac{n}{X\sqrt{q}}\Big) + \epsilon\sum_{n\geq 1}\frac{\overline{\chi(n)}}{\sqrt{n}}V\Big(\frac{nX}{\sqrt{q}}\Big)
\end{aligned}$$

where in the last line we used thee formula $\Lambda(1/2) = q^{1/2}\pi^{-1/4}\Gamma(1/4)L(1/2,\chi)$.

f) One can prove that $|V(y)| \leq e^{-2\pi y}$ for all $y > 0$. Deduce from this that for any real number $\epsilon > 0$, the estimate

$$L(1/2,\chi) = O(q^{1/4+\epsilon})$$

holds, where the implied constant depends on ϵ.

[Hint: take $X = 1$ above, then split the sums in the parts where $n \leq q^{1/2+\epsilon}$ and the remainder.]

Solution:

Let $\varepsilon > 0$. Taking $X = 1$ in the equation in part e) together with noting that $|\chi(n)| \leq 1$ and $|\varepsilon| = 1$, we have

$$\begin{aligned} L(1/2, \chi) &\ll \sum_{n \geq 1} \frac{\chi(n)}{\sqrt{n}} e^{-2\pi \frac{n}{\sqrt{q}}} \ll \sum_{n \geq 1} \frac{1}{\sqrt{n}} e^{-2\pi \frac{n}{\sqrt{q}}} \\ &= \sum_{n \leq q^{\frac{1}{2}+\varepsilon}} \frac{1}{\sqrt{n}} e^{-2\pi \frac{n}{\sqrt{q}}} + \sum_{n \leq q^{\frac{1}{2}+\varepsilon}} \frac{1}{\sqrt{n}} e^{-2\pi \frac{n}{\sqrt{q}}} \end{aligned}$$

As suggested in the hint, we will consider the two sums above separately. For the first sum we have

$$\sum_{n \leq q^{\frac{1}{2}+\varepsilon}} \frac{1}{\sqrt{n}} e^{-2\pi \frac{n}{\sqrt{q}}} \leq \sum_{n \leq q^{\frac{1}{2}+\varepsilon}} \frac{1}{\sqrt{n}} \ll \int_1^{q^{\frac{1}{2}+\varepsilon}} \frac{1}{\sqrt{x}} dx \ll q^{\frac{1}{4}+\varepsilon}$$

Using a geometric sum, we obtain for the second sum

$$\sum_{n > q^{\frac{1}{2}+\varepsilon}} \frac{1}{\sqrt{n}} e^{-2\pi \frac{n}{\sqrt{q}}} \leq \sum_{n > q^{\frac{1}{2}+\varepsilon}} e^{-2\pi \frac{n}{\sqrt{q}}} \ll \frac{e^{-2\pi q^{\varepsilon}}}{1 - e^{-2\pi/\sqrt{q}}} = O(1)$$

Hence

$$L(1/2, \chi) = O(q^{\frac{1}{4}+\varepsilon})$$