Recall : f is a holomorphic modular form on $SL_2(\mathbb{Z})$ of weight k (even) integer if $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ z \in \mathbb{H} \Rightarrow f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

Many aspects can be generalized.

$\boxed{(1)}$ Replace $SL_2(\mathbb{Z})$ by another subgroup of $SL_2(\mathbb{R})$: for $\Gamma \subset SL_2(\mathbb{R})$, look for $f : \mathbb{H} \longrightarrow \mathbb{C}$ with

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

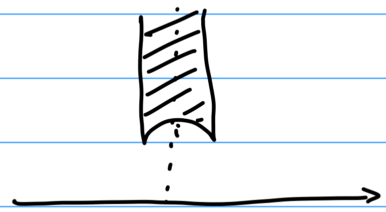
for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Γ should be "not too big", i.e. discrete

Γ ——— "not too small", i.e. it has a fundamental domain D_Γ which has finite (hyperbolic) area i.e.

$$\int_{D_\Gamma} \frac{dx\,dy}{y^2} < +\infty$$

[Ex. $\Gamma = SL_2(\mathbb{Z})$



$$\int_D \frac{dx\,dy}{y^2} \leq \int_1^\infty \int_{-1/2}^{1/2} dx\, \frac{dy}{y^2} = \underbrace{\int_1^\infty \frac{dy}{y^2}}_{< +\infty}$$
]

Note that the measure $\frac{dx\,dy}{y^2}$ is $SL_2(\mathbb{R})$-invariant, so the $\int_{D_\Gamma} \frac{dx\,dy}{y^2}$ will be independent of the choice of D_Γ.

In the interpretation of $\mathbb{H}$ as the hyperbolic plane, $\frac{dx\,dy}{y^2}$ is the (invariant) measure on $\mathbb{H}$.

Ex. (1) If $\Gamma \subset SL_2(\mathbb{Z})$ has finite index, one can take

$$D_\Gamma = \bigcup_{g \in SL_2(\mathbb{Z})/\Gamma} g\, D_{SL_2(\mathbb{Z})}$$

so $$\int_{D_\Gamma} \frac{dx\,dy}{y^2} = [SL_2(\mathbb{Z}) : \Gamma] \int_D \frac{dx\,dy}{y^2} < +\infty.$$

Examples include "congruence subgroups" such as

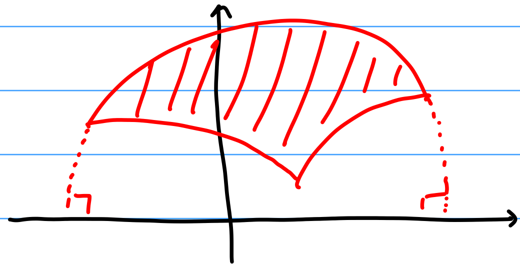
$\left(\begin{array}{c} q \geq 1 \\ \text{integer} \\ \text{"level"} \end{array}\right)$ $$\Gamma(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod q \right\}$$

$(\text{index } |SL_2(\mathbb{Z}/q\mathbb{Z})|)$

or $$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod q \right\}$$

$$\left[[SL_2(\mathbb{Z}) : \Gamma_0(q)] = q \prod_{p \mid q} \left(1 + \frac{1}{p}\right) \right]$$

(2) There are many subgroups Γ of $SL_2(\mathbb{R})$ which are discrete and such that $\Gamma \backslash \mathbb{H}$ is compact with the quotient topology (so D_Γ is compact)

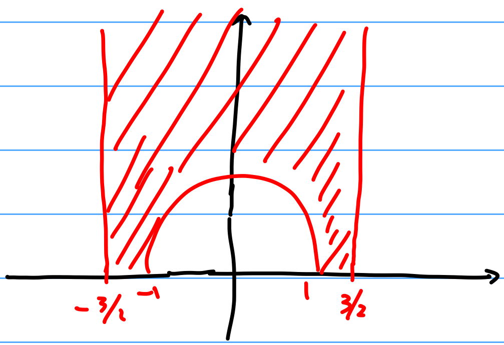


(Not easy to describe, constructed using definite quaternion algebras)

(3) Let Γ = subgroup of $SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

[compare with $SL_2(\mathbb{Z})$, generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$]

One can show that a fundamental domain is



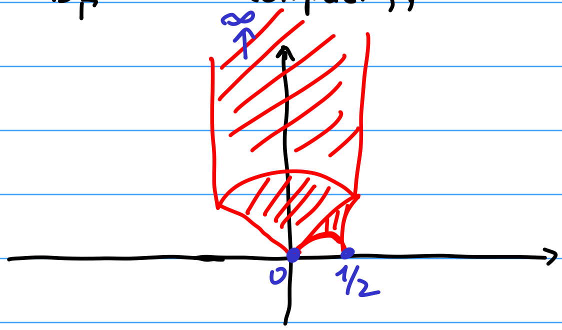
and $\displaystyle\int_{D_\Gamma} \frac{dx\,dy}{y^2} = +\infty$

so the group is "too small".

Define modular forms on a $\Gamma \subset SL_2(\mathbb{R})$ discrete with D_Γ of finite area as in the case of $SL_2(\mathbb{Z})$.

(i) There may non-zero modular forms of odd weight if $-1 \notin \Gamma$ (eg. $\Gamma = \Gamma(q)$ if $q \geq 3$)

(ii) To define forms holomorphic "at ∞" / cusp forms, we have to take into account that there sometimes more ways to go to ∞ than for $SL_2(\mathbb{Z})$ (and sometimes no way to do so if D_Γ is compact); D_Γ might look like this:



where we have different "cusps".

For each cusp there will be an associated $\tilde{f}$ which tells us if f is holomorphic / vanishes at ∞.

$\Rightarrow$ one defines

$$M_k(\Gamma) = \{ f \ \Gamma\text{-modular, wt. } k, \text{ holomorphic at all cusps} \}$$
$$\cup$$
"cusp forms" $M_k^\circ(\Gamma) = \{ f \in M_k(\Gamma) \mid f \text{ vanishes at all cusps} \}$
$$\shortparallel$$
$$S_k(\Gamma)$$

Facts:
- all these spaces are finite-dimensional.
- one can define (for $k \geq 3$) Eisenstein and Poincaré series (at each cusp)
- one can compute the Fourier expansions (with "generalized" Kloosterman sums)
- the dimension of $M_k(\Gamma)$ is harder to compute; for instance $\dim_{\mathbb{C}} S_2(\Gamma)$ is a very important invariant of $\Gamma \backslash \mathbb{H}$ and very often it is large (e.g. $\lim_{q \to \infty} \dim_{\mathbb{C}} S_2(\Gamma_0(q)) = +\infty$).

Q: What are such modular forms good for?

It turns out that many interesting functions $\mathbb{H} \xrightarrow{f} \mathbb{C}$ are modular (of some weight) but only for a subgroup of $SL_2(\mathbb{Z})$.

Ex. (Theta series)

Q = positive definite quadratic form in r variables with integral coefficients (e.g. $x_1^2 + \dots + x_r^2$)

$(z \in \mathbb{H})$ $$\Theta_Q(z) = \sum_{x \in \mathbb{Z}^r} e(Q(x) z) \qquad \left(\text{e.g. } \left(\sum_{n \in \mathbb{Z}} e(n^2 z)\right)^r\right)$$

If r is even then Θ_Q is always modular of weight $k = \frac{r}{2}$ for some subgroup $\Gamma_Q \subset SL_2(\mathbb{Z})$ (congruence subgroup of level related to the discriminant of Q)

Understanding the modular forms in $M_{r/2}(\Gamma_Q)$ leads to information on the Fourier expansion
$$\Theta_Q(z) = \sum_{n \geq 0} r_Q(n) e(nz)$$
where
$$r_Q(n) = \sum_{\substack{x \in \mathbb{Z}^r \\ Q(x) = n}} 1 .$$

One can check that $\Theta_Q \in M_{\frac{r}{2}}(\Gamma_Q)$, so one gets an expression
$$\Theta_Q = \underbrace{E_Q}_{\text{"Eisenstein series"}} + \underbrace{C_Q}_{\text{cusp form}}$$
and so a formula for $r_Q(n)$ follows by computing Fourier expansions:

→ for E_Q it is often explicit (analogues of σ_{k-1})

→ for C_Q it is far from explicit, but one can show that the coefficients of C_Q are of smaller order of magnitude compared to E_Q

Conclusion: asymptotic formula for $r_Q(n)$ as $n \to \infty$

2 Replacing $(cz+d)^k$ by something more general

A motivation comes from looking at

$(z \in \mathbb{H})$ $\qquad \Theta_Q(z) = \sum_{x \in \mathbb{Z}^r} e(Q(x)z)$

for Q in an odd number of variables (eg. $x_1^2 + x_2^2 + x_3^2$) where one gets Hol Θ_Q is "modular" of weight $\frac{r}{2} \in \frac{1}{2} + \mathbb{N}$ but with $(cz+d)^k$ replaced by a more complicated expression involving arithmetic information.

Ex. $\qquad \theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z)$ is of weight $\frac{1}{2}$ (Jacobi)

$$\eta(z) = e\left(\frac{z}{24}\right) \prod_{n \geq 1} (1 - e(nz))$$
(Dedekind)

is of weight $\frac{1}{2}$, but (with a different "multiplier system")

"Half-integral weight" modular forms have some similar properties as integral weight forms, but also very striking differences: in particular, the Fourier coefficients of such forms are very far from being multiplicative.

For instance, a theorem of Gauss gives an expression for

$$r_{x_1^2+x_2^2+x_3^2}(n)$$

as a function (essentially) of the class number of $\mathbb{Q}(\sqrt{-n})$.

$\hookrightarrow$ very mysterious, not multiplicative

Another change of the factor $(cz+d)^k$ is milder: for $\Gamma \subset SL_2(\mathbb{R})$ (k integer), let $\chi : \Gamma \longrightarrow S^1$ group morphism and look at $f : \mathbb{H} \longrightarrow \mathbb{C}$ s.t.

$\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ z \in \mathbb{H}\right)$ $\qquad f(gz) = \chi(g)(cz+d)^k f(z)$ $\qquad$ "nebentypus"

Ex. $\qquad \Gamma = \Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod q \right\} \subset SL_2(\mathbb{Z})$

For $\tilde{\chi}$ a Dirichlet character modulo q, define

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tilde{\chi}(d).$$

<u>Notation:</u> $M_k(\Gamma, \chi)$, $S_k(\Gamma, \chi)$

$\boxed{3}$ <u>Changing "regularity" of f</u>

The space $C^\infty(\Gamma\backslash\mathbb{H})$ makes sense but it is ∞ dimensional. To get smaller spaces (which generate a big subspace) one looks at functions $f: \mathbb{H} \longrightarrow \mathbb{C}$, smooth, Γ-invariant which are eigenfunctions of the <u>hyperbolic Laplace operator</u>: for some $\lambda \in \mathbb{C}$,

$$z \in \mathbb{H} \Rightarrow \Big(\underbrace{-y^2\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)}\Big)(z) = \lambda f(z)$$

[Maass forms; Selberg]

hyperbolic Laplace operator, invariant under $SL_2(\mathbb{R})$

With growth conditions at ∞ (cusps), or integrability properties w.r.t. $\frac{dx\,dy}{y^2}$, one gets analogues of M_k, S_k and one can show:

- the corresponding spaces are finite dimensional
- there are only at most countably many λ, necessarily $\lambda \geqslant 0$, such that $S_\lambda(\Gamma) \neq \{0\}$

<u>Big difference with holomorphic forms</u>: except for <u>very special</u> cases with $\lambda = 1/4$, no single "explicit" Maass cusp form $\neq 0$ is known! (For $SL_2(\mathbb{Z})$ or $\Gamma_0(q)$, Selberg proved that there are ∞-ly many linearly independent, but without writing a generating set.)

<u>Maass</u>: related to real quadratic fields

$\boxed{4}$ Changing $SL_2(\mathbb{R})$ and $\mathbb{H}$

$\left(\begin{array}{ll} \text{Siegel} \longrightarrow & Sp_{2g}(\mathbb{Z}) \\ \text{Langlands} & 1960\text{'s} \end{array}\right)$

In Exercise Sheet 1 one sees how to go from

$$f : \mathbb{H} \longrightarrow \mathbb{C}$$

modular to

$$\varphi : SL_2(\mathbb{R}) \longrightarrow \mathbb{C}$$
$$g \longmapsto f(g \cdot i)$$

This φ is then invariant / transforms in a controlled way by multiplying by $\gamma \in SL_2(\mathbb{Z})$ on the left:

$$\varphi(\gamma g) = \varphi(g) \qquad (\text{or related to } \varphi(g))$$

This suggest to replace $SL_2(\mathbb{R})$ by another group G with a (discrete) subgroup $\Gamma \subset SL_2(\mathbb{R})$ and looking directly at

$$\varphi : G \longrightarrow \mathbb{C}$$

s.t. $\varphi(\gamma g) = \varphi(g)$ (or is related to it).

<u>Ex.</u> $G = SL_n(\mathbb{R})$, $n \geq 2$ $\qquad$ $G = GL_n(\mathbb{R})$

$\Gamma = SL_n(\mathbb{Z})$ $\qquad$ $(GL_n(\mathbb{Z}))$

$(g \geq 1)$ $G = Sp_{2g}(\mathbb{R})$

$\Gamma = Sp_{2g}(\mathbb{Z})$

Imposing conditions of growth / differential equations will generalize all the previous constructions.

<u>Langlands</u>: if G is a "reductive group" (examples as above) and if we have certain group homomorphisms

$${}^{L}G_1 \longrightarrow {}^{L}G_2$$

"L-group", defined by Langlands $\left[\begin{array}{l} \underline{\text{Ex.}} \;\; G = GL_n(\mathbb{R}) \\ \longrightarrow {}^{L}G = GL_n(\mathbb{C}) \end{array}\right]$

then there is a "natural" way to map modular forms for G_1 to those for G_2. (related to L-functions of modular forms)

("Langlands functoriality")