Recall: There exists a modular function $j : \mathbb{H} \longrightarrow \mathbb{C}$ holomorphic on $\mathbb{H}$, with a simple pole at ∞ with residue 1, such that it induces a bijection

$$SL_2(\mathbb{Z}) \backslash \mathbb{H} \longrightarrow \mathbb{C}$$

In fact, $j = \lambda \dfrac{E_4^3}{\Delta}$ for some $\lambda \in \mathbb{C}^\times$, $\Delta = E_4^3 - E_6^2 \in M_{12}^0$.

Corollary. Any modular function f which is meromorphic everywhere, including at ∞, is of the form

$$f = Q(j)$$

for some $Q \in \mathbb{C}(X)$.

Proof. (1) We may assume f is holomorphic on $\mathbb{H}$: indeed, f has finitely many poles in $\mathbb{H}$ (modulo $SL_2(\mathbb{Z})$), since there is no singularity of $\tilde{f}$ in a neighborhood of 0, which translates to no singularity for $\mathrm{Im}(z) \geq y_0$ for some $y_0 > 0$.

Say $\alpha_1, \dots, \alpha_m$ are these poles; then the function

$$f \prod_{i=1}^{m} (j - j(\alpha_i))^{-v_{\alpha_i}(f)}$$

is a modular function, and is holomorphic on $\mathbb{H}$.

(2) Since Δ has a simple zero at ∞, there is an integer $n \geq 0$ such that $g = \Delta^n f$ is a modular form of weight $12n$ holomorphic including at ∞.

So $g \in M_{12n}$.

Then g is of the form

$$\sum_{\substack{i,j \\ 4i+6j=12n}} \alpha_{ij} E_4^i E_6^j, \quad \alpha_{ij} \in \mathbb{C}$$

For any i, j with $4i + 6j = 12n$, we have $3 | i$, $2 | j$.

Then

$$f = \frac{g}{\Delta^n} = \sum_{i,j} \alpha_{ij} \frac{(E_4^3)^{i/3}(E_6^2)^{j/2}}{\Delta^{i/3}\Delta^{j/2}}$$

so it suffices to check that $\frac{E_4^3}{\Delta}$ and $\frac{E_6^2}{\Delta}$ are rational functions of j. But

$$\frac{E_4^3}{\Delta} = \lambda^{-1} j$$

$$\frac{E_6^2}{\Delta} = \frac{E_4^3 - \Delta}{\Delta} = \lambda^{-1} j - 1$$

so the proof is finished.

$\square$

<u>Note</u>. This should be compared to the fact that any function $f : \mathbb{C} \longrightarrow \mathbb{C}$ which is meromorphic everywhere (incl. at ∞) is a rational function.