4 – Simultaneous diagonalization

Example – Let $q = 1$ and $k = 12$. Then for $n \geq 1$, $T(n)$ is a linear map

$$S_{12}(1) = M_{12}^{0} \longrightarrow S_{12}(1)$$

Since $\dim S_{12}(1) = 1$, generated by the Δ-function

$$\Delta(z) = e(z) \prod_{n \geq 1} (1 - e(nz))^{24}$$

$$= \sum_{n \geq 1} \tau(n)\, e(nz)$$

it follows that for each n, Δ is an eigenvector of $T(n)$. Since $\tau(1) = 1$, we deduce from the previous discussion that

$$T(n)\, \Delta = \lambda_n\, \Delta$$

with $\tau(n) = \lambda_n\, \tau(1) = \lambda_n$.

In particular, it follows

(1) that $\tau(mn) = \tau(m)\, \tau(n)$ if $(m, n) = 1$ (multiplicativity of the Hecke operators)

(2) more generally, consider the general multiplication formula

$$\tau(m)\, \tau(n) = \sum_{d \mid (m,n)} d^{11}\, \tau\left(\frac{mn}{d^2}\right).$$

Apply this to $m = p^{\nu}$, $\nu \geq 0$, $n = p$, for p prime: we get

$$\tau(p)\, \tau(p^{\nu}) = \tau(p^{\nu+1}) + p^{11}\, \tau(p^{\nu-1}).$$

It follows (by multiplying out) that

$$\underset{\tau(1)}{\overset{\shortparallel}{1}} + \tau(p)\,p^{-s} + \tau(p^2)\,p^{-2s} + \dots$$

$$= \frac{1}{1 - \tau(p)\,p^{-s} + p^{11-2s}}$$

So we obtain

$$L(\Delta, s) = \sum \frac{\tau(n)}{n^s} = \prod \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$$

for $\mathrm{Re}(s)$ large enough, and this admits an analytic continuation to an entire function with

$$(2\pi)^{-s}\,\Gamma(s)\,L(\Delta, s) = (2\pi)^{s-12}\,\Gamma(12-s)\,L(\Delta, 12-s)$$

This allows us also to "understand" the Ramanujan conjecture $|\tau(n)| \leq d(n)\,n^{11/2}$: write

$$1 - \tau(p)\,X + p^{11}X^2 = (1-\alpha_p X)(1-\beta_p X).$$

Then
$$\begin{cases} \tau(p) = \alpha_p + \beta_p \\ \alpha_p \beta_p = p^{11} \end{cases}$$

<u>Conj.</u> (Ramanujan) / <u>Theorem</u> (Deligne)

We have $|\alpha_p| = |\beta_p| = p^{11/2}$.

By multiplicativity we deduce

$$\tau(p^k) = \alpha_p^k + \alpha_p^{k-1}\beta_p + \dots + \alpha_p\beta_p^{k-1} + \beta_p^k$$

$$\Rightarrow |\tau(p^k)| \leq (k+1)\,p^{11k/2} = d(p^k)\,p^{11k/2}$$

$$\Rightarrow |\tau(n)| \leq d(n)\,n^{11/2} \text{ for all } n.$$

Question: does this nice situation extend to other spaces of modular forms, of dimension ≥ 2?

The first answer is the following.

Theorem (Hecke)

For $k \geq 2$, there exists a unique basis of $S_k(1) = M_k^0$ whose elements are eigenfunctions of all $T(n)$ for $n \geq 1$, and each element has 1st Fourier coeff. $= 1$.

[In particular, one can really say that $S_k(1)$ has a "canonical" basis (unordered).]

The key points are the following two lemmas.

Lemma 1. Let E be a (finite-dimensional) Hilbert space and $(u_i)_{i \in I}$ a family of commuting normal operators ($u_i u_i^* = u_i^* u_i$). There exists an orthonormal basis of E s.t. all u_i are diagonal in this common basis.

Proof. Exercise: the key points are

(i) every normal $u \in End(E)$ is diagonalizable in orthonormal basis

(ii) if u, v commute then v defines an endomorphism of every eigenspace of u.

$\square$

Lemma 2. Each $T(n)$, $n \geq 1$, defines a <u>normal</u> endomorphism of $S_k(1)$ for the Petersson inner product

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} \, y^k \frac{dx\,dy}{y^2}$$

where D is a fundamental domain for $SL_2(\mathbb{Z})$.

(In fact, the proof will show that $T(n)$ is self-adjoint, but this is somewhat special.)

Proof. It is enough to prove that $T(p)^* = T(p)$ for p prime, since the multiplication formula will then imply that $T(n)^* = T(n)$ for all n.

SubLemma. For $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n$, we have

$$\int_D (f_1|_k g) \, \overline{f_2} \, y^k \frac{dx\,dy}{y^2} = \int_D f_1 \, \overline{(f_2|_k g')} \, y^k \frac{dx\,dy}{y^2}$$

where $g' = \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$.

Proof We observe that $f_1|_k g \in S_k(\Gamma(n))$, where $\Gamma(n)$ ~~[illegible]~~ is the finite index subgroup of $SL_2(\mathbb{Z})$ ~~[illegible]~~ of matrices $\equiv 1 \bmod n$ (direct computation).

Then

$$\int_D f_1|_k g \cdot \overline{f_2} \, y^k \frac{dx\,dy}{y^2} = \frac{1}{[SL_2(\mathbb{Z}):\Gamma(n)]} \int_{D_n} f_1|_k g \, \overline{f_2} \, y^k \frac{dx\,dy}{y^2}$$

$$= \frac{1}{[SL_2(\mathbb{Z}):\Gamma(n)]} \int_{D_n} f_1 \, \overline{f_2|_k g'} \, y^k \frac{dx\,dy}{y^2}$$

where D_n is a fundamental domain for $\Gamma(n)$, and we made the change of variable $\begin{cases} w = gz = \frac{az+b}{d} \\ z = \frac{dw-b}{a} \end{cases}$

This is
$$\int_D f_1 \, \overline{f_2|_k g'} \, y^k \frac{dx\,dy}{y^2}$$
by redoing the first step backwards.

$\square$

Now write
$$< f_1|T(p), f_2 > = p^{k/2-1} \int_D \sum_{g \in \Delta_p} f_1|_k g(z) \, \overline{f_2(z)} \, y^k \frac{dx\,dy}{y^2}$$

$$= p^{\frac{k}{2}-1} \sum_{g \in \Delta_p} \int_D f_1|_k g \, \overline{f_2} \, y^k \frac{dx\,dy}{y^2}$$

Sublemma
$$= p^{k/2-1} \sum_{g \in \Delta_p} \int f_1 \, \overline{f_2|_k g'} \, y^k \frac{dx\,dy}{y^2}$$

~~By invariance under translations we can replace~~
$$\cancel{f_2|_k g' = f_2|_k \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} = f_2|_k \begin{pmatrix} d & b \\ 0 & a \end{pmatrix}}$$

Now note that for $n = p$ the matrices involved are either $g = \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$, $0 \le b < p$

or $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$

and in both cases we can find γ, γ' in $SL_2(\mathbb{Z})$ such that

$$\gamma' g' = g \gamma$$

(if $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, take $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma' = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}$

if $g = \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$, take $\gamma = \begin{pmatrix} -b & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma' = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}$]

Since f_1, f_2 are $SL_2(\mathbb{Z})$-modular, it follows that

$$\int_D f_1 \overline{f_2|_k g'} \; y^k \frac{dx\,dy}{y^2}$$
$$= \int_D f_1 \overline{f_2|_k g} \; y^k \frac{dx\,dy}{y^2}$$

and then $\langle f_1 | T(p), f_2 \rangle = \langle f_1, T(p) | f_2 \rangle$, which gives the result.

□

Proof of Hecke's Theorem – Lemmas 1 and 2 give the simultaneous diagonalization. ~~What remains is to check that the basis is unique. Let $(f_i)_{i\in I}$ be one such basis.~~

Claim 1 – If $f \neq 0$ is a simultaneous eigenfunction of all $T(n)$, then $a_f(1) \neq 0$.

Indeed recall that $T(n) f = \lambda_n f$ implies that

$$a_f(n) = \lambda_n \, a_f(1)$$

so $a_f(1) = 0$ implies $f = 0$.

So we can replace any element of a basis of simultaneous eigenfunctions by a multiple to ensure $a_f(1) = 1$ for these.

<u>Claim 2.</u> If $(f_i)_{i \in I}$ is a basis of simultaneous eigenfunctions and f is a simult. eigenfunction then it is proportional to one f_i.

This gives the stated uniqueness.

To see this, write

$$f = \sum_{i \in I} \alpha_i f_i$$

Then, writing $T(n) f = \lambda_n f$, $T(n) f_i = \lambda_{n,i} f_i$, we get

$$\sum_{i \in I} \lambda_n \alpha_i f_i = \lambda_n f = T(n) f = \sum_{i \in I} \alpha_i \lambda_{n,i} f_i$$

and so

$$\lambda_n \alpha_i = \lambda_{n,i} \alpha_i \quad , \quad \forall i \ \forall n$$

Take i s.t. $\alpha_i \neq 0$; then we get $\lambda_n = \lambda_{n,i} \ \forall n$, but since

$$a_f(n) = \lambda_n \, a_f(1)$$

this means that $f = a_f(1) f_i$.

$\square$