We can illustrate very concretely the complexity of the Hecke basis and of Hecke eigenvalues: the next theorem suggest strongly that these have no simple expressions.

First, we recall a definition.

<u>Def.</u> Let X be a compact metric space. Let (μ_n) be a sequence of (Radon) measures on X and μ a Radon measure on X. One says that μ_n converges to μ $\left(\begin{smallmatrix}\text{weakly}\\ \text{in law}\end{smallmatrix}\right)$ if

$$\forall f : X \longrightarrow \mathbb{C} \text{ continuous}, \quad \int f\, d\mu = \lim_{n\to\infty} \int f\, d\mu_n. \qquad (*)$$

We will apply this to X a compact interval in $\mathbb{R}$, and to measures μ_n of the type

$$\mu_n = \sum_{x \in X_n} \alpha_x \,\delta_{\Theta_n(x)}$$

where

$$\Theta_n : X_n \longrightarrow X$$

is a map from a finite set to X, and $\alpha_x \geqslant 0$.
[In particular, when $X_n \neq \emptyset$ and $\alpha_x = \frac{1}{|X_n|}$ for all x, the resulting definition is that of μ-equidistribution:

$$\int f\, d\mu = \lim_{n\to\infty} \frac{1}{|X_n|} \sum_{x \in X_n} f(\Theta_n(x))$$

<u>Ex.</u> (Weyl) - Let $\alpha \in \mathbb{R} - \mathbb{Q}$, $X = \mathbb{R}/\mathbb{Z}$ and

$$X_n = \{1, \ldots, n\}$$

$$\Theta_n(j) = j\alpha \bmod 1 \quad (= \text{fractional part of } j\alpha)$$

Proposition. (Weyl Criterion)

To have $\mu_n \to \mu$ it is enough to prove $(*)$ for a family of functions $f: X \to \mathbb{C}$ which are continuous, span a dense subset of $\mathcal{C}(X) = \{f: X \to \mathbb{C}, f \text{ continuous}\}$, and with $1 \in \mathcal{F}$.

Sketch of proof. $(*)$ extends linearly to the subspace generated by $\mathcal{F}$; for arbitrary $f \in \mathcal{C}(X)$, and $\varepsilon > 0$, find g in this space with

$$\sup_{x \in X} |f(x) - g(x)| \leq \varepsilon.$$

Then

$$\begin{aligned} \left| \int f \, d\mu_n - \int f \, d\mu \right| &\leq \int |f - g| \, d\mu_n \\ &+ \int |f - g| \, d\mu \\ &+ \left| \int g \, d\mu_n - \int g \, d\mu \right| \\ &\leq \sup |f - g| \left(\mu_n(X) + \mu(X) \right) \\ &+ \left| \int g \, d\mu_n - \int g \, d\mu \right| \end{aligned}$$

By $(*)$ for $f = 1 \in \mathcal{F}$ we have $\mu_n(X) \to \mu(X)$ so $(\mu_n(X))$ is bounded. So for some $C \geq 0$

$$\left| \int f \, d\mu_n - \int f \, d\mu \right| \leq C\varepsilon + \left| \int g \, d\mu_n - \int g \, d\mu \right|.$$

Since $(*)$ holds for g, we can find n_0 s.t.

$$\left| \int g \, d\mu_n - \int g \, d\mu \right| \leq \varepsilon$$

for all $n \geq n_0$, and then
$$\left|\int f\,d\mu_n - \int f\,d\mu\right| \leq (C+1)\varepsilon.$$
$\square$

<u>Ex.</u> For $j\alpha \bmod 1$, take μ = Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ and $\mathcal{F} = \{e_h \mid h \in \mathbb{Z}\}$ with
$$e_h(x) = e^{2i\pi h x}$$
Then $1 \in \mathcal{F}$ and $\mathcal{F}$ spans the space of trigonometric polynomials, which is dense in $\mathcal{C}(\mathbb{R}/\mathbb{Z})$. So Weyl's Criterion applies. Now $(*)$ is immediate for $h=0$ because μ_n and μ are probability measures. Then for $h \neq 0$
$$\int e_h\,d\mu_n = \frac{1}{n}\sum_{j=1}^{n} e^{2i\pi j\alpha h}$$
$$= \frac{1}{n}\, e^{2i\pi h\alpha}\, \frac{1-e^{2i\pi n\alpha h}}{1-e^{2i\pi h\alpha}}$$
since <u>$\alpha h \notin \mathbb{Z}$</u> (because $\alpha \notin \mathbb{Q}$), so
$$\left|\int e_h\,d\mu_n\right| \leq \frac{1}{n}\,\frac{2}{|1-e^{2i\pi h\alpha}|} \xrightarrow[n\to\infty]{} 0$$
So we have equidistribution. In particular for f continuous on $\mathbb{R}/\mathbb{Z}$, we get for instance
$$\int_{\mathbb{R}/\mathbb{Z}} f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(j\sqrt{2} \bmod 1)$$

Now we go back to modular forms...

<u>Theorem</u>. (Serre, Sarnak, K.-Saha-Tsimerman)

Let k be s.t. $S_k(1) \neq \{0\}$, so $\mathcal{H}_k \neq \emptyset$.

Let p be a prime number.

For $f \in \mathcal{H}_k$, let $\lambda_f(p) = \dfrac{a_f(p)}{p^{(k-1)/2}}$, where $a_f(p)$ is the p-th Fourier coefficient of f.

Define

$$\mu_k = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_k} \frac{1}{\|f\|^2} \delta_{\lambda_f(p)}$$

which is a measure on $\mathbb{R}$, supported on a compact interval since $|\lambda_f(p)| = O(\sqrt{p})$ (by the bound on Fourier coefficients of cusp forms proved earlier).

<u>Then</u> μ_k converges weakly as $k \to \infty$ to the so-called <u>Sato-Tate measure</u>

$$\mu_{ST} = \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}}\, \mathbb{1}_{[-2,2]}\, dx.$$

<u>Cor.</u> The set $\{\lambda_f(p) \mid f \in \mathcal{H}_k \text{ for some } k\}$ is dense in $[-2,2]$ (i.e. its closure contains $[-2,2]$).

<u>Proof.</u> We apply the previous discussion, and especially the Weyl Criterion with $\mathcal{F} = \{U_n \mid n \geq 0\}$ where U_n is the n-th Chebychev polynomial (of the second kind), i.e. the unique $U_n \in \mathbb{R}[X]$ such that

$$\frac{\sin((n+1)\theta)}{\sin(\theta)} = U_n(2\cos\theta).$$

Note $U_0 = 1$.

[<u>Ex.</u> $U_0 = 1$, $U_1 = X$, $U_2 = X^2 - 1$, $U_3 = X^3 - 2X$, ...)

Lemma – (1) The functions U_n span a dense subspace of $\mathcal{C}(I)$ for any compact interval $I \subset \mathbb{R}$.
(2) The functions U_n form an orthonormal basis of $L^2([-2,2], \mu_{ST})$.

Proof (1) It is elementary that U_n is a polynomial of degree n, so this follows from the Stone–Weierstrass approximation theorem.

(2) It is elementary that the U_n are ortho-normal, using the change of variable

$$x = 2\cos\theta, \qquad 0 \leq \theta \leq \pi$$

for which μ_{ST} becomes $\frac{2}{\pi}\sin^2\theta \, d\theta$, so

$$\int_{-2}^{2} U_n U_m \, \mu_{ST}(x) = \frac{2}{\pi}\int_0^{\pi} \sin^2(\theta) \frac{\sin((n+1)\theta)\sin((m+1)\theta)}{\sin^2(\theta)}$$

To prove that $(U_n)_{n \geq 1}$ is a basis requires a bit more work. One can

(i) reduce to Fourier series after the same change of variable (assume $\langle f | U_n \rangle = 0 \ \forall n$, etc).

(ii) use more advanced tools from representation theory, specifically the Peter–Weyl Theorem for representations of $SU_2(\mathbb{C})$ and the classification of the irreducible representations of $SU_2(\mathbb{C})$, which turn out to have "character" given by the (U_n). $\square$

So we need to compute

$$\int U_n \, d\mu_k = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_k} \frac{1}{\|f\|} U_n(\lambda_f(p))$$

<u>Lemma 2.</u> For p prime and $n \geq 1$, we have

$$U_n(\lambda_f(p)) = \lambda_f(p^n).$$

<u>Proof.</u> Recall that

$$\sum_{n \geq 0} T(p^n) X^n = \frac{1}{1 - T(p)X + p^{k-1}X^2}$$

which applied to $f \in \mathcal{H}_k$ gives

$$\sum a_f(p^n) X^n = \frac{1}{1 - a_f(p)X + p^{k-1}X^2}$$

and

$$\sum \lambda_f(p^n) X^n = \frac{1}{1 - \lambda_f(p)X + X^2}.$$

Expand $1 - \lambda_f(p)X + X^2 = (1 - \alpha_p X)(1 - \beta_p X)$

and write $0 \neq \alpha_p = e^{i\theta_p}$, so $\beta_p = e^{-i\theta_p}$

and $\lambda_f(p) = 2\cos(\theta_p)$ [note θ_p might not be real]. Expand in geometric series to get

$$\begin{aligned}
\lambda_f(p^n) &= \sum_{j=0}^{n} \alpha_p^j \beta_p^{n-j} \\
&= e^{-in\theta_p} + e^{-i(n-2)\theta_p} + \dots \\
&\quad + e^{i(n-2)\theta_p} + e^{in\theta_p} \\
&= e^{-in\theta_p} \cdot \frac{e^{2i\theta_p(n+1)} - 1}{e^{2i\theta_p} - 1} \\
&= \frac{\sin((n+1)\theta_p)}{\sin(\theta_p)} = U_n(2\cos\theta_p) = U_n(\lambda_f(p)).
\end{aligned}$$

$\square$

We are left then with

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_k} \frac{\lambda_f(p^n)}{\|f\|^2}.$$

But $\mathcal{F}_k^* = \left\{ \frac{f}{\|f\|} \right\}$ is an orthonormal basis of $S_k(1)$. So, we get

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{\tilde{f} \in \mathcal{F}_k^*} \frac{a_{\tilde{f}}(p^n)}{p^{n(k-1)/2}} \frac{1}{\|f\|}$$

$$= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{\tilde{f} \in \mathcal{F}_k^*} \frac{a_{\tilde{f}}(p^n)\, a_{\tilde{f}}(1)}{\sqrt{p^{n(k-1)}\, 1^{k-1}}}$$

We recognize the left-hand side of the Petersson formula for

$$\sum_{f \in \mathcal{F}_k^*} \frac{a_f(m)\, a_f(n)}{(mn)^{(k-1)/2}}$$

(where $\mathcal{F}_k^*$ could be any orthonormal basis of $S_k(1)$; cf. page 3 of the third online lecture).

This gives

$$\int U_n \, d\mu_k = \delta(p^n, 1) + \frac{2\pi}{i^k} \sum_{c \geq 1} \frac{1}{c} S(p^n, 1, c)\, J_{k-1}\left(\frac{4\pi p^{n/2}}{c}\right).$$

in terms of Kloosterman sums and Bessel functions.

Our goal is to prove that

$$\int U_n \, d\mu_k \longrightarrow \int U_n \, \mu_{ST} = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases}$$

(by orthogonality of the Chebychev polynomials), and so this follows from

$$\lim_{k\to\infty} \sum_{c\geq 1} \frac{1}{c} S(p^n, 1; c) J_{k-1}\left(\frac{4\pi p^{n/2}}{c}\right) = 0.$$

This can be proved directly (because for a given c, we have $J_{k-1}\left(\frac{4\pi p^{n/2}}{c}\right) \xrightarrow[k\to\infty]{} 0$) but here is a slightly different "softer" proof: going back to the proof of the Petersson formula, we have in fact

$$\int U_n \, \mu_k = p^n\text{-th Fourier coefficient of } P_{1,k}$$

More generally:

<u>Lemma</u>. For $m, n \geq 1$, we have

$$\lim_{k\to\infty} (m\text{-th Fourier coefficient of } P_{n,k}) = \delta(m,n).$$

This obviously gives the result.

<u>Proof</u>. We have

$$a_m(P_{n,k}) = \int_0^1 P_{n,k}(x+i)\, e(-m(x+i))\, dx$$

by definition, and

$$P_{n,k}(z) = \sum_{g\in \overline{N}\backslash SL_2(\mathbb{Z})} \frac{e(ng.z)}{(cz+d)^k}$$

Note that $\left| \dfrac{e(n g\cdot z)}{(cz+d)^k} \right| \leq |cz+d|^{-k}$

and $|cz+d|^2 \geqslant c^2 y^2 \geqslant c^2$ if $z = x+i$.

So for $c \neq 0$ we get

$$\left| \frac{e(n g\cdot z)}{(cz+d)^k} \right| \xrightarrow[k\to\infty]{} 0$$

uniformly for $\begin{cases} g \in \overline{N}\backslash SL_2(\mathbb{Z}), \ c \neq 0 \\ z = x+i \text{ with } 0 \leq x \leq 1. \end{cases}$

For $c = 0$, we have $g = \mathrm{Id}$ (class mod $\overline{N}$)

so

$$\frac{e(n g z)}{(cz+d)^k} = e(nz)$$

~~Since moreover~~ Since moreover $\left| \dfrac{e(n g z)}{(cz+d)^k} \right| \leq |cz+d|^{-4}$

for $k \geqslant 4$, which defines a convergent series, it follows by dominated convergence that

$$P_{n,k}(z) \xrightarrow[k\to\infty]{} e(nz)$$

and then that

$$\int_0^1 P_{n,k}(x+i)\, e(-m(x+i))\, dx \xrightarrow[k\to\infty]{} \int_0^1 e((n-m)(x+i))\, dx = \delta(m,n)$$

$\square$

<u>Remarks.</u> (1) If we replace the measure μ_k by

$$\frac{1}{|\mathcal{H}_k|} \sum_{f \in \mathcal{H}_k} \delta_{\lambda_f(p)}$$

(which may seem more natural), there is a similar result but the limit is not μ_{ST} anymore: it is

$$\mu_{Pl,p} = \frac{p+1}{\pi} \frac{\sqrt{1 - x^2/4}}{\left(\sqrt{p} + \frac{1}{\sqrt{p}}\right)^2 - x^2} \, \mathbb{1}_{[-2,2]} \, dx$$

(Serre 1997)

(2) One consequence of the result is that "usually" $\lambda_f(p) \in [-2, 2]$. In fact, this is always true:

<u>Th.</u> (Deligne, "Ramanujan-Petersson conjecture")

For $f \in \mathcal{H}_k$, p prime we have $|\lambda_f(p)| \leq 2$

i.e. $\quad |a_f(p)| \leq 2 p^{\frac{k-1}{2}}$.

For all n, we get

$$|a_f(n)| \leq d(n) \, n^{\frac{k-1}{2}}$$

(3) What about <u>fixing</u> some f and looking at $\lambda_f(p)$ as $p \to \infty$? This is a much harder problem, but the answer is similar:

<u>Th.</u> (Clozel, Harris, Shepherd-Barron, Taylor - 2007; "Sato-Tate conjecture")

As $p \to \infty$, the $\lambda_f(p)$ become μ_{ST}-distributed:

for all $f: [-2,2] \longrightarrow \mathbb{C}$ continuous, we have

$$\int f \, d\mu_{ST} = \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} f(a_f(p))$$