3. Hecke operators

When Γ is "arithmetic" (in a certain precise sense) it was discovered by Hecke that modular forms for Γ have "extra" symmetries arising from the existence of certain linear maps on spaces like $M_k(\Gamma)$. This applies in particular to $\Gamma = \Gamma_0(q)$ and the resulting <u>Hecke operators</u> are essential in the modern theory of modular forms.

The definition of Hecke operators can be given in different ways. We follow the (slightly "ad hoc") approach of Iwaniec, but suggest to also look at Serre's discussion ((Ch. VII, § 5).

The basic idea is a form of averaging again.

<u>Lemma</u>. Let $n \geq 1$ be an integer. Let

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; ad = n, \quad 0 \leq b < d \right\}$$

Then Δ_n parameterizes the $SL_2(\mathbb{Z})$-orbits of the subset

$$G_n = \left\{ g \in M_2(\mathbb{Z}) \;\middle|\; \det(g) = n \right\}$$

for the $SL_2(\mathbb{Z})$-action by left-multiplication: there is a disjoint union

$$G_n = \bigcup_{g \in \Delta_n} SL_2(\mathbb{Z})\, g.$$

Proof. (1) The union is disjoint: if

$$\gamma_1 g_1 = \gamma_2 g_2 , \qquad \begin{array}{l} \gamma_i \in SL_2(\mathbb{Z}) \\ g_i \in \Delta_n \end{array}$$

then we get

$$\gamma g_1 = g_2 , \qquad \gamma = \gamma_2^{-1} \gamma_1$$

so an equality

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$$

$$ru - st = 1 \qquad\qquad \begin{array}{l} a_i d_i = n \\ 0 \leq b_i < d_i \end{array}$$

which immediately gives $t = 0$ (since $a_1 \neq 0$)

Then $n = a_2 d_2 = r u a_1 d_1 \Rightarrow r = u = 1$

so $\begin{cases} a_2 = a_1 \\ b_2 = b_1 \end{cases}$

and then we get $s = 0$ because $0 \leq b_i < d$.

(2) Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n$. We first bring it to upper-triangular form: note that

$$\gamma = \frac{c}{(a,c)} , \qquad \delta = -\frac{a}{(a,c)}$$

are coprime and

$$\gamma a + \delta c = 0 .$$

Extend $(\gamma \ \delta)$ to $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$; then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$$

for some integers a_1, b_1, d_1. We have $a_1 d_1 = n$, and up to multiplying by -1 if needed, we can assume $a_1, d_1 \geq 1$.

Then there is some $u \in \mathbb{Z}$ s.t.

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 + u d_1 \\ 0 & d_1 \end{pmatrix}$$

satisfies $\quad 0 \leq b_1 + u d_2 < d_1$, and then

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_n$$

so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\, \Delta_n .$$

$\square$

<u>Definition</u> - Fix k, $q \geq 1$ and a Dirichlet character χ mod q. Let

$$P_k = \{ f : \mathbb{H} \to \mathbb{C} \mid f \text{ is } 1\text{-periodic} \}.$$

Define $\quad T_n : P_k \longrightarrow P_k \quad$ [depending on k, q, χ]

by

$$T(n) f(z) = n^{\frac{k}{2} - 1} \sum_{g \in \Delta_n} (f|_k\, g)(z)\, \chi(a)$$

for $z \in \mathbb{H}$, where for one puts

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$$

one puts

$$(f|_k\, \gamma)(z) = (\det \gamma)^{k/2} (cz + d)^{-k} f(\gamma z)$$

<u>Note</u>_ $\quad T(n) f(z) = n^{k/2 - 1} \sum_{ad = n} \chi(a)\, n^{k/2} d^{-k} f\left(\frac{az+b}{d}\right)$

$$= \frac{1}{n} \sum_{ad=n} \chi(a)\, a^k f\left(\frac{az+b}{d}\right)$$

Lemma – $T(n)$ is well-defined: if f is 1-periodic, so is the function $T(n)f$ defined above.

Proof – By definition

$$T_n(f)|_k \begin{pmatrix}1&1\\0&1\end{pmatrix} = n^{\frac{k}{2}-1} \sum_{g\in\Delta_n} f|_k\, g\, |_k \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

Claim: For $g\in\Delta_n$, there is a unique $v\in\mathbb{Z}$ and $g'\in\Delta_n$ such that

$$g\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}1&v\\0&1\end{pmatrix} g'$$

and moreover $g \longmapsto g'$ is bijective, $\chi(a)=\chi(a')$

This claims gives

$$T_n(f)|_k \begin{pmatrix}1&1\\0&1\end{pmatrix} = n^{k/2-1} \sum_{g'\in\Delta_n} \chi(a')\, \underbrace{f|_k\begin{pmatrix}1&v\\0&1\end{pmatrix}}_{=f \text{ by periodicity}} |_k\, g'$$

$$= T_n(f)$$

so $T_n(f)$ is periodic.

The claim itself follows from the Lemma on page 10: $g\begin{pmatrix}1&1\\0&1\end{pmatrix}\in G_n$ so has a unique expression

$$\gamma g', \quad \begin{cases}\gamma\in SL_2(\mathbb{Z})\\ g'\in\Delta_n\end{cases}$$

and just by examination, one sees that in fact $\gamma=\begin{pmatrix}1&v\\0&1\end{pmatrix}$ for some v:

$$\begin{pmatrix}a&b\\0&d\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}a&a+b\\0&d\end{pmatrix}$$

$\cancel{\begin{pmatrix}1&v\\0&1\end{pmatrix}\begin{pmatrix}a'&b'\\0&d'\end{pmatrix}}$ and $\begin{pmatrix}1&v\\0&1\end{pmatrix}\begin{pmatrix}a'&b'\\0&d'\end{pmatrix} = \begin{pmatrix}a&b'+vd\\0&d\end{pmatrix}$

so we can find ν such that $\begin{cases} a+b = b' + \nu d \\ 0 \le b' < d \end{cases}$.

$\square$

This allows us to compute directly the Hecke operators action on Fourier expansions, which makes certain aspects of the theory quite transparent.

<u>Corollary</u>. Suppose $f \in \mathcal{M}_k$ satisfies

$$f(z) = \sum_{m\ge 0} a(m)\, e(mz)$$

Then

$$T_n f(z) = \sum_{m \ge 0} \left(\sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, a\left(\frac{mn}{d^2}\right)\right) e(mz)$$

(Note in particular the surprising symmetry between m and n in this formula.)

<u>Proof</u>. By definition

$$T_n f(z) = n^{\frac{k}{2}-1} \sum_{g \in \Delta_n} (f|_k g)(z)\, \chi(a)$$

$$= n^{k/2-1} \sum_{\substack{ad=n \\ a,d\ge 1}} \chi(a) \sum_{0\le b<d} n^{k/2}\, d^{-k}\, f\left(\frac{az+b}{d}\right)$$

$$= \frac{1}{n} \sum_{\substack{ad=n \\ a,d\ge 1}} \chi(a) \sum_{0\le b<d} a^k \sum_{m\ge 0} a(m)\, e\left(\frac{m(az+b)}{d}\right)$$

We exchange the order of these sums to start with m.

This gives

$$\sum_{m\geq 0} a(m) \frac{1}{n} \sum_{\substack{ad=n\\ a,d\geq 1}} \chi(a)\, a^{k}\, e\left(\frac{maz}{d}\right) \underbrace{\sum_{0\leq b<d} e\left(\frac{mb}{d}\right)}_{=\begin{cases}0, & d\nmid m\\ d, & d\mid m\end{cases}}$$

$$= \sum_{m\geq 0} a(m) \sum_{\substack{ad=n\\ d\mid m}} \chi(a)\, a^{k-1}\, e\left(\frac{amz}{d}\right)$$

$$= \sum_{\substack{ad=n\\ a,d\geq 1}} \chi(a)\, a^{k-1} \sum_{l\geq 0} a(ld)\, e(alz)$$

$$= \sum_{r\geq 0} e(rz) \underbrace{\sum_{\substack{al=r\\ ad=n}} \chi(a)\, a^{k-1}\, a(ld)}_{\sum_{a\mid (r,n)} \chi(a)\, a^{k-1}\, a\left(\frac{rm}{d^2}\right)}$$

$\square$

<u>Corollary</u>. Assume again $f \in P_k$ has Fourier expansion $f(z) = \sum_{m\geq 0} a_f(m)\, e(mz)$.

(1) $a_{T_n(f)}(0) = \left(\sum_{d\mid n} d^{k-1}\chi(d)\right) a_f(0)$

(2) $a_{T_n(f)}(1) = a_f(n)$

(3) If $T_n(f) = \lambda f$ then either $a_f(0) = 0$ or $\lambda = \sum_{d/n} d^{k-1} \chi(d)$.

(4) If $T_n(f) = \lambda f$ then
$$a_f(n) = \lambda a_f(1).$$

Proof. This follows immediately from the computation of the Fourier expansion.

□

Theorem. The Hecke operators give linear maps
$$M_k(q,\chi) \longrightarrow M_k(q,\chi)$$
$$S_k(q,\chi) \longrightarrow S_k(q,\chi)$$

Proof. The idea is similar to the case of periodic functions: given $\gamma \in \Gamma_0(q)$ and $g \in \Delta_n$, $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, we have
$$\underset{\gamma}{\gamma} \underset{g}{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} = \underset{g'}{\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}} \gamma'$$
for a unique $\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \Delta_n$ and $\gamma' \in SL_2(\mathbb{Z})$.

The point is then that:

(i) if $(a,q)=1$ then $(a',q)=1$

(ii) $\gamma' \in \Gamma_0(q)$

(iii) $\overline{\chi(\gamma)}\, \chi(a) = \chi(a')\, \overline{\chi(\gamma')}$

(iv) The map $g \longmapsto g'$ is a bijection of $\{g \in \Delta_n / (a,q)=1\}$

Indeed, if we assume this, we get for $f \in M_k(q,\chi)$

and $\gamma \in \Gamma_0(q)$:

$$T(n) f|_k \gamma = n^{\frac{k}{2}-1} \sum_{\substack{g \in \Delta_n \\ \parallel \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ (a,q)=1}} \chi(a)\, f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} |_k \gamma$$

$$= n^{k/2-1} \sum_{g \in \Delta_n, (a,q)=1} \chi(a)\, f|_k \gamma' |_k g'$$

$$= n^{k/2-1} \sum_{\substack{g \in \Delta_n \\ (a,q)=1}} \underbrace{\chi(a)\, \chi(\gamma')}_{\substack{\parallel \\ \chi(a')\,\chi(\gamma)}} f|_k g'$$

$$= \chi(\gamma)\, n^{k/2-1} \sum_{\substack{g' \in \Delta_n \\ (a',q)=1}} \chi(a')\, f|_k g'$$

$$= \chi(\gamma)\, T(n) f.$$

The holomorphy / vanishing at all cusps is checked by using the fact that the "cusps are permuted".

Now we check the properties above:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \gamma = \gamma' \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

Write $\gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, $\gamma' = \begin{pmatrix} r' & s' \\ t' & u' \end{pmatrix}$; then

$$\begin{pmatrix} ra + tb & * \\ dt & * \end{pmatrix} = \begin{pmatrix} r'a' & * \\ t'a' & * \end{pmatrix}$$

Since $q | t$ we have $ra \equiv r'a' \bmod q$. ~~[illegible]~~

If $(a,q) = 1$, then since r, r' are coprime to q

we get $(a', q) = 1$. Then from

$$t'a' = dt \equiv 0 \ (q) \qquad (\gamma \in \Gamma_0(q))$$

we deduce $t' \equiv 0 \ (q)$ so $\gamma' \in \Gamma_0(q)$. Finally

$$ar \equiv a'r' \mod q$$

gives

$$\underbrace{\overline{\chi(\gamma)}}_{\chi(r)} \chi(a) = \chi(a') \overline{\chi(\gamma')} .$$

Finally $g \mapsto g'$, on $\{ g \in O_n \mid (a, q) = 1 \}$ is injective, hence bijective

$\square$

Proposition. The Hecke operators on $M_k(q, \chi)$ (or on P_k) satisfy the relations

$$T(m)\, T(n) = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, T\!\left(\frac{mn}{d^2}\right)$$

for $m, n \geq 1$. In particular

$$T(m)\, T(n) = T(n)\, T(m)$$

and $T(m) T(n) = T(mn)$ if $(m, n) = 1$.

Proof. By definition

$$mn\, T(m) T(n) f = \sum_{\substack{a_1 d_1 = m \\ a_2 d_2 = n}} \chi(a_1 a_2)\, (a_1 a_2)^k \sum_{\substack{b_1 \\ b_2}} f\big|_k \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$$

$$= \sum_{\substack{a_1 d_1 = m \\ a_2 d_2 = n}} \chi(a_1 a_2)\, (a_1 a_2)^k \sum_{\substack{b_1 \bmod d_1 \\ b_2 \bmod d_2}} f\big|_k \begin{pmatrix} a_1 a_2 & b \\ 0 & a_2 d_2 \end{pmatrix}$$

with $b = a_1 b_2 + b_1 d_2$.

If m and n are coprime, then b runs uniquely over (representatives of) $\mathbb{Z}/d_1d_2\mathbb{Z}$ as b_1, b_2 vary. Using periodicity, which representative is used in $f|_k\begin{pmatrix} a_1a_2 & b \\ 0 & d_1d_2 \end{pmatrix}$ <u>does not matter</u> so

$$mn\, T(m)T(n)f = \sum_{\substack{a_1d_1=m \\ a_2d_2=n}} \chi(a_1a_2)(a_1a_2)^k \sum_{b \bmod d_1d_2} f|_k\begin{pmatrix} a_1a_2 & b \\ 0 & d_1d_2 \end{pmatrix}$$

Moreover, since we assume $(m,n)=1$, we see that the pairs $(a_1, d_1), (a_2, d_2)$ [$a_1d_1=m$, $a_2d_2=n$] are in bijection with (a,d) such that $ad=mn$. So

$$mn\, T(m)\,T(n) = mn\, T(mn)$$

in this case.

The general case is a bit more involved. The entry $b = d_2b_1 + a_1b_2$, for d_2, a_1 varying is always a multiple of $\delta = (a_1, d_2)$. So are a_1a_2 and d_1d_2. Thus we split the sum according to δ and replace a_1 by δa_1, d_2 by δd_2

$$\sum_{\delta | (m,n)} \chi(\delta)\delta^k \sum_{\substack{a_1d_1=m/\delta \\ a_2d_2=n/\delta \\ (a_1,d_2)=1}} \chi(a_1a_2)(a_1a_2)^k \sum_{\substack{b_1 (d_1) \\ b_2 (\delta d_2)}} f|_k\begin{pmatrix} a_1a_2 & b \\ 0 & d_1d_2 \end{pmatrix}$$

with again $b = d_2b_1 + a_1b_2$.

<u>Claim</u>. As $b_1\ (d_1)$ and $b_2\ (\delta d_2)$ vary, the value $b = d_2b_1 + a_1b_2$ ranges δ times over

every class modulo $d_1 d_2$ (because b determines b_2 mod d_2 and b_1 mod d_1).

So the sum becomes

$$\sum_{\delta \mid (m,n)} \chi(\delta)\, \delta^{k+1} \sum_{\substack{a_1 d_1 = m/\delta \\ a_2 d_2 = n/\delta \\ (a_1, d_2) = 1}} (a_1 a_2)^k \chi(a_1 a_2) \sum_{b \bmod d_1 d_2} f|_k \begin{pmatrix} a_1 a_2 & b \\ 0 & d_1 d_2 \end{pmatrix}$$

Now observe that the pairs (a_1, d_1), (a_2, d_2) correspond bijectively to pairs with $\begin{Bmatrix} a = a_1 a_2 \\ d = d_1 d_2 \end{Bmatrix}$ such that $ad = \frac{mn}{\delta^2}$ and $(a_1, d_2) = 1$.

Thus ~~the sum~~ is

$$mn\, T(m)\, T(n) = \sum_{\delta \mid (m,n)} \chi(\delta)\, \delta^{k+1} \sum_{ad = \frac{mn}{d^2}} a^k \chi(a) \sum_{b \bmod d} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$\Rightarrow T(m)\, T(n) = \sum_{\delta \mid (m,n)} \chi(\delta)\, \delta^{k-1} \frac{1}{\left(\frac{mn}{\delta^2}\right)} \sum_{ad = \frac{mn}{d^2}} a^k \chi(a) \sum_{b\,(d)} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= \sum_{\delta \mid (m,n)} \chi(\delta)\, \delta^{k-1}\, T\!\left(\frac{mn}{d^2}\right) f.$$

$\square$