Chapter IV

Theta functions

1. Introduction

Theta functions are functions on $\mathbb{H}$ which give rise to automorphic forms which are quite different from those we already know how to construct. They are related to integral positive-definite quadratic forms, and their modularity properties give rise to many properties for the representations of integers by such quadratic forms, e.g. by sums of squares.

We begin with a general definitions. Let $r \geq 1$ integer, Q be a positive-definite quadratic form given by

$$Q(x) = x A \, {}^t x$$

for some integral symmetric matrix $A \in M_r(\mathbb{Z})$. In other words

$$Q(x_1, \ldots, x_r) = \sum_{i=1}^{r} a_{ii} x_i^2 + 2 \sum_{1 \leq i < j \leq r} a_{ij} x_i x_j$$

for $x = (x_j) \in \mathbb{R}^r$, with $Q(x) > 0$ whenever $x \neq 0$.

We say that Q is <u>even</u> if $a_{ii} \equiv 0 \ (2)$ for all

; in which case $Q(x) \equiv 0 \mod 2$ for all $x \in \mathbb{Z}^r$.

The "basic" theta function attached to Q is the function

$$\theta_Q : \mathbb{H} \longrightarrow \mathbb{C}$$

defined by

$$\theta_Q(z) = \sum_{m \in \mathbb{Z}^r} e\left(\frac{1}{2} Q(m) z\right).$$

<u>Ex.</u> $Q(x) = x^2$ is the simplest quadratic form (with $r=1$); then

$$\theta(z) = \sum_{m \in \mathbb{Z}} e\left(\frac{1}{2} m^2 z\right)$$

$$= \sum_{m \in \mathbb{Z}} e^{i\pi m^2 z}.$$

For concrete applications, we will also consider special cases of the more general functions

$$\theta_Q(z; f) = \sum_{m \in \mathbb{Z}^r} e\left(\frac{1}{2} Q(m) z\right) f(m)$$

for some function $f : \mathbb{Z}^r \longrightarrow \mathbb{C}$.

We will show (most of) the following result:

<u>Theorem.</u> (cf. Iwaniec, Cor. 10.7)

Let Q be as above with r even. Let $N \geq 1$ be an integer such that $NA^{-1} \in M_r(\mathbb{Z})$. [e.g. $N = |\det(A)|$ will do]

(1) We have $\theta_Q \in M_{r/2}(\Gamma(4N))$, i.e.

$$\theta_Q(\gamma z) = (cz+d)^{r/2} \theta_Q(z)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 4N$

(2) If moreover both A, NA^{-1} have even diagonal coefficients, then

$$\theta_Q \in M_{r/2}(\Gamma_0(N), \chi)$$

where $\chi(\gamma) = \left(\frac{\det A}{d}\right)$ is a real Dirichlet character modulo $\prod_{p \mid \det A} p$.

This only addresses the case of an <u>even</u> number of variables. There is indeed a genuine important difference with ~~the~~ the case r odd, and although some statement of similar nature is valid, it involves "half-integral weight" modular forms, which are quite delicate. Instead of developping this part of the story in general, we just look now at the simplest case, $r = 1$.

2 _ "One-variable" theta functions and Dirichlet L-functions

We will look at the following theta functions for $Q(x) = x^2$:

$$\theta(z; \chi) = \sum_{m \in \mathbb{Z}} \chi(m)\, e\left(\frac{m^2 z}{2}\right)$$

for some ~~$\theta(z, \chi)$~~ Dirichlet character $\chi \bmod q \geq 1$ and

$$\theta(z; a) = \sum_{\substack{m \in \mathbb{Z} \\ m \equiv a (q)}} e\left(\frac{m^2 z}{2}\right)$$

for $a \in \mathbb{Z}$.

<u>Lemma 1</u> - $\theta(z;\chi)$ and $\Theta(z;a)$ are well-defined holomorphic functions on $\mathbb{H}$ such that

$$\theta(z+2;\chi) = \theta(z;\chi)$$

$$\Theta(z+2;a) = \Theta(z;a)$$

<u>Proof</u> - Note that for $z = x+iy \in \mathbb{H}$, we get

$$\left| e\left(\frac{m^2 z}{2}\right)\right| = e^{-\pi m^2 y},$$

and hence the series

$$\sum_{m \in \mathbb{Z}} e\left(\frac{m^2 z}{2}\right)$$

then converges locally absolutely, and therefore its sum is a holomorphic function on $\mathbb{H}$.

The 2-periodicity is also obvious since

$$e\left(\frac{m^2(z+2)}{2}\right) = e\left(\frac{m^2 z}{2}\right)$$

for all $m \in \mathbb{Z}$.

$\square$

<u>Note</u>: We would get 1-periodicity if using $Q(x) = 2x^2$: this explains why Q even is slightly easier to work with.

The crucial property of theta function is basically contained in the next lemma.

Lemma 2 – ("Jacobi transformation formula")

Let $y > 0$ be a real number. We then have

$$\theta(iy; a) = \frac{1}{q\sqrt{y}} \sum_{x \bmod q} e\left(\frac{ax}{q}\right) \theta\left(\frac{-1}{q^2(iy)}; x\right)$$

and if χ is primitive

$$\theta(iy; \chi) = \frac{\tau(\chi)}{q\sqrt{y}}\, \theta\left(-\frac{1}{q^2(iy)}, \overline{\chi}\right)$$

where

$$\tau(\chi) = \sum_{x \bmod q} \chi(x)\, e\left(\frac{x}{q}\right)$$

is the Gauss sum.

Recall that χ being primitive refers to the fact that it cannot be defined to a modulus $< q$; if q is prime, this is the same as saying that χ is not the trivial character.

Proof – We have

$$\theta(iy; a) = \sum_{\substack{m \in \mathbb{Z} \\ m \equiv a (q)}} e^{-\pi m^2 y}$$

$$= \sum_{n \in \mathbb{Z}} e^{-\pi (a+nq)^2 y}$$

$$= \sum_{n \in \mathbb{Z}} f(n)$$

with $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(x) = e^{-\pi(qx+a)^2 y}.$$

Since f has rapid deccay and is smooth (it is a so-called Schwatz function), this sum can be transformed using the Poisson summation formula (cf. Online Lecture on Poincaré Series, p° 5):

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{h\in\mathbb{Z}} \hat{f}(h)$$

where

$$\hat{f}(t) = \int_{\mathbb{R}} f(x)\, e(-xt)\, dx$$

for $t \in \mathbb{R}$ [the integral always exist since f is integrable]. The value of $\hat{f}$ here is classical: we get by definition

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-\pi(xq+a)^2 y}\, e(-xt)\, dx$$

We write $xq + a = u$, $q\,dx = du$, to get

$$\hat{f}(t) = \frac{1}{q}\int_{\mathbb{R}} e^{-\pi u^2 y}\, e\left(-t\left(\frac{u-a}{q}\right)\right) du$$

$$= \frac{1}{q}\, e\left(\frac{at}{q}\right) \int_{\mathbb{R}} e^{-\pi u^2 y}\, e\left(-\frac{tu}{q}\right) du$$

Then $u^2 y = v^2$ with $v = \sqrt{y}\,u$, $dv = \sqrt{y}\,du$, gives

$$\hat{f}(t) = \frac{1}{q}\, e\left(\frac{at}{q}\right) \frac{1}{\sqrt{y}} \int_{\mathbb{R}} e^{-\pi v^2}\, e\left(\frac{-t}{q\sqrt{y}} v\right) dv$$

$$= \frac{1}{q}\, e\left(\frac{at}{q}\right) \frac{1}{\sqrt{y}}\, \hat{g}\left(\frac{t}{q\sqrt{y}}\right)$$

where $g(x) = e^{-\pi x^2}$. The point is that g has an easy Fourier transform to remember:

<u>Lemma 3</u>. $\hat{g} = g$

<u>Proof</u>. By differentiation under $\int$ (easily justified here since g, g' both decay exponentially fast at $\pm\infty$) ~~one checks that~~ followed by integration by parts, one gets the differential equation

$$\hat{g}' = -2\pi t \hat{g}$$

The space of solutions is generated by g, so $\hat{g} = \lambda g$ for some $\lambda \in \mathbb{C}$. Putting $t = 0$ gives

$$\lambda = \hat{g}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1,$$

by a classical computation.

$\square$

Coming back to $\theta(iy; a)$, we get now

$$\theta(iy;a) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{h\in\mathbb{Z}} \hat{f}(h)$$

$$= \sum_{h\in\mathbb{Z}} \frac{1}{q}\, e\left(\frac{ah}{q}\right) \frac{1}{\sqrt{y}}\, e^{-\pi \frac{h^2}{q^2 y}}$$

$$= \frac{1}{q\sqrt{y}} \sum_{h\in\mathbb{Z}} e\left(\frac{ah}{q}\right) e\left(\frac{\pi h^2}{2q^2(iy)}\right)$$

$$= \frac{1}{q\sqrt{y}} \sum_{x \bmod q} e\left(\frac{ax}{q}\right) \sum_{\substack{h\in\mathbb{Z} \\ h\equiv x (q)}} e\left(\frac{\pi h^2}{2q^2(iy)}\right)$$

(only depends on h mod q)

$$= \frac{1}{q\sqrt{y}} \sum_{x (q)} e\left(\frac{ax}{q}\right) \theta\left(\frac{-1}{q^2(iy)}; x\right)$$

Let now χ be a primitive character mod q.
Then

$$\theta(z;\chi) = \sum_{m\in\mathbb{Z}} \chi(m)\, e\left(\frac{zm^2}{2}\right)$$

(only depends on $a\ (q)$)

$$= \sum_{a \bmod q}' \chi(a)\, \theta(z;a).$$

~~(cas précédent)~~ $\cancel{= \frac{1}{q\sqrt{x}} \sum_{x \bmod q} \sum_{a \bmod q} \chi(a)\, e\left(\frac{ax}{q}\right) \theta(}$

For $z = iy$, the previous case then gives

$$\theta(iy;\chi) = \sum_{a \bmod q} \chi(a) \frac{1}{q\sqrt{y}} \sum_{x \bmod q} e\left(\frac{ax}{q}\right) \times \theta\left(-\frac{1}{q^2(iy)}; x\right)$$

$$= \frac{1}{q\sqrt{y}} \sum_{x \bmod q} \theta\left(-\frac{1}{q^2(iy)}; x\right) \times \sum_{a \bmod q} \chi(a)\, e\left(\frac{ax}{q}\right).$$

Now a property of <u>primitive</u> characters (easy to check for q prime, $\chi \neq 1$) is that

$$\sum_{a \bmod q} \chi(a)\, e\left(\frac{ax}{q}\right) = \tau(\chi)\, \overline{\chi(x)}$$

for <u>all</u> x mod q [it is always true for $(x,q)=1$, but not always if $(x,q)\neq 1$ and χ non primitive].

Therefore we get

$$\theta(iy;\chi) = \frac{1}{q\sqrt{y}}\, \tau(\chi) \sum_{x \bmod q} \overline{\chi}(x)\, \theta\left(-\frac{1}{q^2(iy)}; x\right)$$

$$= \frac{1}{q\sqrt{y}} \; \theta\left(-\frac{1}{q^2(iy)} \; ; \; \overline{\chi}\right)$$

as claimed.

□

This has the following very important corollary.

<u>Corollary</u>. Let $q \geq 1$ be an integer and χ a Dirichlet character modulo q which is <u>primitive</u>.

(1) If $q=1$, so $L(s,\chi) = \zeta(s)$, then $\zeta(s)$ has analytic continuation to a meromorphic function on $\mathbb{C}$, with a unique simple pole at $s=1$ with residue 1, and which satisfies

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

(2) If $q \geq 2$ then $L(s,\chi)$ has analytic continuation to an <u>entire</u> function, and moreover the function

$$\Lambda(\chi,s) = \pi^{-(s+t_\chi)/2}\Gamma\left(\frac{s+t_\chi}{2}\right) L(\chi,s)$$

satisfies the <u>functional equation</u>

$$\Lambda(\chi,s) = \frac{1}{i^{t_\chi}} \frac{\tau(\chi)}{\sqrt{q}} q^{\frac{1}{2}-s} \Lambda(\overline{\chi},1-s)$$

where

$$t_\chi = \begin{cases} 0, & \chi(-1)=1 \\ 1, & \chi(-1)=1 \end{cases}$$

(so $\chi(-1) = (-1)^{t_\chi}$)

To be precise, this will be proved for χ "even", i.e. for $t_\chi = 0$; the "odd" case requires a slightly different theta function, which we will just briefly describe.

<u>Proof</u>. We assume first that $q \geq 2$ and $t_\chi = 0$. Then we use the elementary formula

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty e^{-\pi n^2 x}\, x^{\frac{s}{2}-1}\, dx$$

which is valid for $n \geq 1$ and $\mathrm{Re}\left(\frac{s}{2}\right) > 1$, by an elementary change of variable in the definition of $\Gamma\left(\frac{s}{2}\right)$. We multiply by $\chi(n)$ and sum over n. Because of the very quick convergence, there is no issue in exchanging $\sum_{n\geq 1}$ and $\int_{x \in]0,+\infty[}$, so

$$\Lambda(s,\chi) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) L(\chi, s) = \int_0^\infty \Big(\sum_{n\geq 1} \chi(n)\, e^{-\pi n^2 x}\Big)\, x^{s/2}\frac{dx}{x}$$

$$= \frac{1}{2}\int_0^\infty \theta(ix;\chi)\, x^{s/2}\frac{dx}{x},$$

since $\begin{cases} \chi(-n) = \chi(-1)\,\chi(n) = \chi(n), & n \leq -1 \\ \chi(0) = 0 & \text{since } q \geq 2 \end{cases}$

This is the same situation as the Hecke L-function, and we analyze it in the same way:

$$\Lambda(s,\chi) = \frac{1}{2}\left(\int_0^{1/q} (-) + \int_{1/q}^{+\infty} \theta(ix;\chi)\, x^{s/2}\frac{dx}{x}\right)$$

Then

$$\int_0^{1/q} \theta(ix;\chi)\, x^{s/2}\frac{dx}{x} = \int_0^{\frac{1}{q}} \frac{\tau(q)}{q\sqrt{x}}\,\theta\left(\frac{i}{q^2 x}, \overline{\chi}\right) x^{\frac{s}{2}-1}\, dx$$

by the "automorphy" of the theta function. The change of variable $y = \frac{1}{q^2 x}$, $\frac{dy}{y} = \cancel{\ } \frac{dx}{x}$, given

$$\int_0^{1/q} \theta\left(\frac{i}{q^2 x}; \bar{\chi}\right) x^{\frac{s-1}{2}} \frac{dx}{x}$$

$$= \cancel{\ }\int_{1/q}^{+\infty} \theta(iy; \bar{\chi}) \left(\frac{1}{q^2 y}\right)^{\frac{s-1}{2}} \frac{dy}{y}$$

and hence

$$\frac{\tau(q)}{q} \int_0^{1/q} \theta\left(\frac{i}{q^2 x}, \bar{\chi}\right) x^{\frac{s}{2}-1} dx$$

$$= \frac{\tau(q)}{q} \cdot \cancel{\ } \cdot q^{1-s} \int_{\frac{1}{q}}^{+\infty} \theta(iy; \bar{\chi})\, y^{\frac{1-s}{2}} \frac{dy}{y}$$

so that

$$\Lambda(s, \chi) = \frac{1}{2}\left(\int_{\frac{1}{q}}^{+\infty} \theta(iy; \chi)\, y^{\frac{s}{2}} \frac{dy}{y} \right.$$

$$\left. + \left(\frac{\tau(\chi)}{\sqrt{q}}\right) q^{1/2 - s} \int_{\frac{1}{q}}^{+\infty} \theta(iy; \bar{\chi})\, y^{\frac{1-s}{2}} \frac{dy}{y} \right),$$

This gives the analytic continuation, and the functional equation follows after checking that

$$\tau(\chi)\,\tau(\bar{\chi}) = q\,\chi(-1) = q.$$

(Note putting $\frac{\tau(q)}{\sqrt{q}}$ together is justified because $|\tau(q)| = \sqrt{q}$, so this is a complex number of modulus 1.)

For $q=1$, namely $\zeta(s)$, the argument is similar but now

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\int_0^{+\infty}\left(\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} e^{-\pi n^2 x}\right) x^{\frac{s}{2}}\frac{dx}{x}$$

$$= \frac{1}{2}\int_0^{\infty}\left(\theta(ix)-1\right)\, x^{s/2}\frac{dx}{x}.$$

One recombines as before, being careful not to integrate $x^{1/2-1}$ near ∞:

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\Bigg[\int_1^{\infty}\left(\theta(ix)-1\right)\, x^{s/2}\frac{dx}{x} + \int_1^{\infty}\left(\theta(iy)-1\right) y^{\frac{1-s}{2}}\frac{dy}{y}$$

$$+\int_0^1 y^{(1-s)/2}\frac{dy}{y} - \int_0^1 x^{s/2}\frac{dx}{x}\Bigg]$$

$$= \frac{1}{s(s-1)} + \frac{1}{2}\int_1^{\infty}\left(\theta(ix)-1\right)\left(x^{s/2}+x^{\frac{1-s}{2}}\right)\frac{dx}{x}$$

and the result follows [note that $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ has a pole also at $s=0$; but this comes from the simple pole of $\Gamma(s)$ at $s=0$.]

For χ odd $(\chi(-1)=-1)$, this method cannot directly succeed, because then

$$\theta(ix;\chi) = \sum_{n\in\mathbb{Z}} \chi(n)\, e^{-\pi n^2 x}$$

$$= 0 \qquad \left(\chi(-n) = -\chi(n)\right)$$

However, one can still proceed using another θ function, namely

$$\theta_1(z;\chi) = \sum_{m\in\mathbb{Z}} m\chi(m)\, e\left(\frac{m^2 z}{2}\right).$$

One proves (by a similar application of the Poisson formula as before) that

$$\theta_1(ix;\chi) = \frac{\tau(\chi)}{iq^2 x^{3/2}}\,\theta_1\left(-\frac{1}{q^2(ix)};\bar{\chi}\right)$$

(i.e it is "of weight $\frac{3}{2}$"), and then the argument is similar, the additional $\frac{1}{i}$ and $\frac{1}{x^{3/2}}$ instead of $\frac{1}{\sqrt{x}}$ end up in the different form of the functional equation.

□

Corollary - The analytic continuation of ζ satisfies

$$\zeta(-2) = \zeta(-4) = \dots = \zeta(-2k) = 0$$

for $k\geq 1$.

Proof. $\Gamma\left(\frac{s}{2}\right)$ has poles at $0, -2, -4, \dots$ but

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

has no pole at $-2, -4, -6, \dots$

□

3 - The basic transformation formula in general

We will now prove (most of) the basic Theorem stated on page 7. It will be seen that it is a sophisticated generalization of the basic 1-variable argument. The main question is probably: how does one achieve modularity for __all__ of $\Gamma(4N)$, when it seems that Poisson summation is more restricted (and $\Gamma(4N)$, in general, is not just generated by translation and inversion).

__Proposition 1.__ – Let Q be a quadratic form as in Section 1; let $N \geq 1$ be s.t. NA^{-1} has integral coefficients. For $z \in \mathbb{H}$, $x \in \mathbb{C}^n$, we have

$$\sum_{m \in \mathbb{Z}^n} e\left(\frac{1}{2} Q(m+x) z\right) = \frac{1}{\sqrt{\det A}} \left(\frac{i}{z}\right)^{n/2}$$

$$\times \sum_{m \in \mathbb{Z}^n} e\Big(-\frac{1}{2z} \tilde{Q}(m) + \underbrace{{}^t m x}_{m \cdot x}\Big)$$

where $\tilde{Q}$ is the quadratic form (not necessarily integral) associated to A^{-1}.

__Proof__ – We use the more general (in appearance) form

$$\sum_{h \in \mathbb{Z}^n} \hat{\varphi}(h)\, e(h \cdot x) = \sum_{m \in \mathbb{Z}^n} \varphi(m+x)$$

of the Poisson summation formula for $\varphi : \mathbb{R}^n \to \mathbb{C}$ with sufficient decay properties, applied to

$$\varphi(y) = e\left(\frac{z}{2} Q(y)\right).$$

Thus we only need to compute the Fourier transform of this function. This amounts to

Fourier transforms of general (finite-dim) Gaussian random vectors.

<u>Lemma.</u> We have

$$\hat{\varphi}(t) = \int_{\mathbb{R}^n} \varphi(y)\, e(-y\cdot t)\, dy = \frac{1}{\sqrt{\det A}} \left(\frac{i}{z}\right)^{\frac{n}{2}} e\left(-\frac{1}{2}\frac{\tilde{Q}(t}{2z}\right.$$

for $t \in \mathbb{R}^n$.

<u>Proof.</u> By analytic continuation <u>with respect to $z \in \mathbb{H}$</u>, it is enough to prove this when $z = iw$ with $w > 0$, so that $\frac{i}{z} = \frac{1}{w}$.

By linear algebra, we can find a basis in which Q becomes the standard sum of squares: there exists a bijective linear map

$$u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that $Q(y) = \|u(y)\|^2 = \sum_{i=1}^{n} u_i(y)^2$, where $u = (u_1, \dots, u_n)$ for $y \in \mathbb{R}^n$.

Then for $t \in \mathbb{R}^n$ we get

$$\hat{\varphi}(t) = \int_{\mathbb{R}^n} e\left(\frac{z}{2} Q(y)\right) e(-y\cdot t)\, dy$$

$$= \int_{\mathbb{R}^n} e\left(\frac{z}{2} \|u(y)\|^2\right) e(-y\cdot t)\, dy$$

$$\cancel{= \frac{1}{|\det(u)|} \int_{\mathbb{R}^n} \prod_{j=1}^{n} e\left(-\frac{z}{2} v_j^2 - \tilde{u}_j(t\right.}$$

$v = u(y)$

$dv = |\det(u)|\, dy$

$$= \frac{1}{|\det(u)|} \int_{\mathbb{R}^n} e\left(\frac{z}{2}\|v\|^2 - u^{-1}(v)\cdot t\right) dv$$

$$= \frac{1}{|\det(u)|} \int_{\mathbb{R}^n} e\left(\frac{z}{2}\|v\|^2 - v \cdot (u^{-1})^*(t)\right) dv$$

where $(u^{-1})^*$ is the adjoint of u^{-1}. By Fubini's Theorem, this splits as a product

$$\frac{1}{|\det(u)|} \prod_{j=1}^{n} \int_{\mathbb{R}} e\left(\frac{z}{2}v^2 - v\,t_j\right) dv$$

where $(t_j) = ((u^{-1})^*(t))$. This is the one-variable computation, and it gives (by looking at what we did earlier)

$$\hat{\varphi}(t) = \frac{1}{|\det(u)|} \prod_{j=1}^{n} \left(\frac{1}{w}\right)^{1/2} e\left(-\frac{1}{2\,(iw)}\, t_j^2\right)$$

[recall $z = iw$ here]

$$= \frac{1}{|\det(u)|} \left(\frac{i}{z}\right)^{n/2} e\left(-\frac{1}{2z}\, \|(u^{-1})^*(t)\|^2\right).$$

We then need to replace the $\|(u^{-1})^*(t)\|^2$ by the quadratic form $\tilde{Q}$. It is maybe more transparent with matrices: let B be s.t. $u(y) = By$; then ${}^tB\,B = A$ (linear algebra) and

$$\|(u^{-1})^*(t)\|^2 = {}^t({}^tB^{-1}t)\,{}^tB^{-1}t = {}^tt\,(B^{-1}\,{}^tB^{-1})\,t$$
$$= {}^tt\,A^{-1}\,t,$$

□

For further applications, we generalize this to "special" generalized Θ functions.

Def. A <u>Q-spherical polynomial</u> is a polynomial $P \in \mathbb{C}[X_1, \ldots, X_n]$ ~~which~~ which is a linear combination of polynomials of the form

(1) P = constant

(2) P = linear form

(3) $P = ({}^t c\, A X)^{\nu}$ for some integer $\nu \geq 2$ and some c s.t. $c \in \mathbb{C}^n$ and $Q(c) = 0$ ("isotropic vector")

The point of this definition will be apparent later when we use these to prove equidistribution theorems for points on spheres of dim. ≥ 4.

Proposition 2 - Let Q be as before and P a Q-spherical polynomial. Then

$$\tilde{P}(x) = P(A^{-1}x)$$

is $\tilde{Q}$-spherical and

$$\sum_{m \in \mathbb{Z}^n} P(m+x)\, e\left(\frac{z}{2} Q(m+x)\right) = \frac{1}{\deg P} \frac{1}{\sqrt{\det A}} \left(\frac{i}{z}\right)^{k} \times \sum_{m \in \mathbb{Z}^n} \tilde{P}(m)\, e\left(-\frac{\tilde{Q}(m)}{2z} + m \cdot x\right)$$

for $z \in \mathbb{H}$, $x \in \mathbb{C}^n$, where $k = \frac{n}{2} + \nu$, $\nu = \deg(P)$.

Proof. For $P =$ constant, this is Proposition 1. We deduce the general case by differentiating. Precisely, we can assume $P(x) = ({}^t c A x)^\nu$ for $\nu \geq 1$, and some $c \in \mathbb{C}^r$, which is arbitrary for $\nu = 1$ and satisfies $Q(c) = 0$ otherwise. We apply successively to

$$(*) \qquad \sum e\left(\frac{z}{2} Q(m+x)\right) = \frac{1}{|\det A|^{1/2}} \left(\frac{i}{z}\right)^{r/2} \sum e\left(-\frac{\tilde{Q}(m)}{2z} + {}^t m x\right)$$

the differential operator

$$D = \sum_{j=1}^{r} c_j \frac{\partial}{\partial x_j}$$

(operating on the variables (x_j) of x).

Lemma. For all x, we have

$$D\, Q(x) = 2\, {}^t c A x$$

and

$$D^\nu Q(x) = 0 \quad \text{for } \nu \geq 2.$$

Proof. The first part can be checked by writing

$$Q(x) = \sum a_{ii} x_i^2 + 2 \sum_{1 \leq i < j \leq r} a_{ij} x_i x_j$$

so that

$$\frac{\partial Q}{\partial x_h} = 2 \sum_i a_{ih} x_i .$$

The second part is just because D^ν involves only partial derivatives of order ≥ 2, so is a constant, and one finds that this constant is $2Q(c) = 0$. $\square$

Applying D once to the LHS of ⊛ gives then

$$(2i\pi z) \sum_m {}^t c A (m+x) \; e\left(\frac{z}{2} Q(m+x)\right)$$

and on the RHS, applying D^ν for $\nu \geq 1$ gives

$$(2i\pi z)^\nu \sum_m e\left(-\frac{\widetilde{Q}(m)}{2z} + m \cdot x\right) (c \cdot m)^\nu$$

On the LHS again, applying D^2 gives

$$(2i\pi z)^2 \sum_m \left({}^t c A (m+x)\right)^2 e\left(\frac{z}{2} Q(m+x)\right)$$

$$+ (2i\pi z) \sum_m \underbrace{{}^t c A c \cdot m}_{=0} \; e\left(\frac{z}{2} Q(m+x)\right)$$

and similarly for higher powers.

To conclude, note simply that

$$\begin{aligned} (c \cdot m)^\nu &= ({}^t c\, m)^\nu \\ &= ({}^t c\, A\, A^{-1} m)^\nu \\ &= P(A^{-1} m). \end{aligned}$$

$\square$

We continue towards the proof of modularity, for these more general theta functions, namely:

<u>Theorem</u> – Let N be as above, P a Q-spherical polynomial homogeneous of degree ν. We have $\Theta_Q(z; P) \in M_{\nu + \frac{1}{2}}(\Gamma(4N))$; if P is not constant, then $\Theta_Q(z; P)$ is a <u>cusp form</u>.

Proof - We will only go through the proof for

$$Q = x_1^2 + \dots + x_n^2 = \|x\|^2, \quad \begin{cases} \tilde{Q} = Q, \\ P^* = P \end{cases}, N = 1.$$

to avoid complications. We write $\Theta = \Theta_Q(\cdot, P)$.

Claim 1 - $\Theta\left(-\frac{1}{z}\right) = \frac{1}{i^\nu} (-iz)^k \, \Theta(z)$

[Proof. We apply

$$\sum_{m \in \mathbb{Z}^n} P(m+x)\, e\left(\frac{z\|m+x\|^2}{2}\right) = \frac{1}{i^\nu} \left(\frac{i}{z}\right)^k \sum_{m \in \mathbb{Z}^n} \overset{P(m)}{\overset{\shortparallel}{P^*(m)}} \times e\left(-\frac{\|m\|^2}{2z} + m \cdot x\right)$$

for $x = 0$ and replace z by $-\frac{1}{z}$.]

Claim 2 - Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(4)$, with $d \neq 0$.

Then

$$\Theta(\gamma z) = \frac{1}{i^{2\nu}} \frac{1}{d^{n/2}} \overline{\Phi} \, (cz+d)^k \, \Theta(z)$$

where

$$\Phi = \sum_{g \in (\mathbb{Z}/d\mathbb{Z})^n} e\left(\frac{b\|g\|^2}{2d}\right)$$

is a generalized Gauss sum in n variables.

[Proof. Let $\tilde{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$.

Then $\tilde{\gamma} z = \frac{bz-a}{dz-c} = \frac{b}{d} - \frac{1}{d(dz-c)}$, so that

$$\Theta(\tilde{\gamma} z) = \sum_{m \in \mathbb{Z}^n} P(m)\, e\left(\frac{\|m\|^2}{2}\left(\frac{b}{d} - \frac{1}{d(dz-c)}\right)\right)$$

The first exponential $e\left(\frac{b\|m\|^2}{2d}\right)$ only depends on m modulo $2d$, and even modulo d since b is <u>even</u>.

So

$$\Theta(\tilde{\gamma} z) = \sum_{g \in (\mathbb{Z}/d\mathbb{Z})^n} e\left(\frac{b\|g\|^2}{2d}\right) \underbrace{\sum_{\substack{m \in \mathbb{Z}^n \\ m \equiv g (d)}} P(m)\, e\left(\frac{d\|m\|^2}{2d^2(dz-c)}\right)}_{\substack{\text{a } \theta \text{ function for } d\|m\|^2 \\ \text{but in a congruence class}}}$$

A generalization of <u>Claim 1</u> gives

$$\frac{i(c-dz)^h}{i^\nu \sqrt{|d \cdot d|}} \sum_{\ell \bmod d} e\left(\frac{\ell \cdot g}{d}\right) \sum_{\substack{m \in \mathbb{Z}^n \\ m \equiv \ell (d)}} P(m)\, e\left(\frac{\|m\|^2 (dz- }{2d}\right.$$

and hence

$$\theta(\tilde{\gamma} z) = \frac{(i(c-dz))^h}{i^\nu d^{1/2}} \sum_{\ell \bmod d} \Phi(\ell) \times \sum_{m \equiv \ell (d)} P(m)\, e\left(\frac{\|m\|^2 z}{2}\right)$$

where

$$\Phi(\ell) = \sum_{g \bmod d} e\left(\frac{b\|g\|^2 - 2\ell\cdot g - c\|\ell\|^2}{2d}\right)$$

Some elementary substitutions show that $\Phi(\ell) = \Phi(0) = \Phi$, so we have

$$\theta(\tilde{\gamma} z) = \frac{1}{i^\nu} \frac{1}{d^{1/2}} (i(c-dz))^h\, \Phi(z)$$

Now note that $\gamma z = \tilde{\gamma}\left(-\frac{1}{z}\right)$, so

$$\theta(\gamma z) = \frac{1}{i^{v}} \frac{1}{d^{r/2}} \Phi\left(i(c + d/z)\right)^{k} \theta\left(-\frac{1}{z}\right)$$

$$= \frac{1}{i^{2v}} \frac{1}{d^{r/2}} \Phi(cz+d)^{k} \theta(z)$$

by Claim 1 again.

Claim 3. $\Phi = d^{r/2}$

This generalizes the computation of squares of Gauss sums; having r even avoids the need to know the sign of the quadratic Gauss sum; the fact that $d \equiv 1$ (4) also plays a role: the square of the Gauss sum is d instead of id.

This concludes the proof. (The fact that θ is hol. at cusp / cusp form is elementary because we "see" the Fourier expansion)

4. Application: equidistribution of points on spheres

We continue with $Q(x) = \|x\|^2$, r even.

Theorem. (cf. Iwaniec, Th. 11.2)

$$\rho_r(n) = |\{m \in \mathbb{Z}^r \mid \|m\|^2 = n\}|$$

$$= \frac{(2\pi)^k}{\Gamma(k)} n^{r/2 - 1} \sigma_r(n) + O\left(n^{\frac{k}{2} - 1/4 + \varepsilon}\right)$$

if $r \geq 4$ is even, $\varepsilon > 0$, where

$$\sigma_r(n) \asymp 1. \quad \left(= \prod_p \lim_{v \to \infty} \frac{|\{x \in (\mathbb{Z}/p^v\mathbb{Z})^r \mid \|x\|^2 = n \bmod p^v\}|}{p^{v(r-1)}}\right)$$

Now let
$$\mathcal{E}(n) = \left\{ \frac{m}{\sqrt{n}} \mid \|m\|^2 = n, \ m \in \mathbb{Z}^{d} \right\}$$
which are "integral" points on the sphere
$$\mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d \mid \|x\|^2 = 1 \}.$$
We know there are "lots" of points. How are they distributed?

We study this using the Weyl Criterion for equidistribution which means that we must understand
$$\frac{1}{\rho_d(n)} \sum_{x \in \mathcal{E}(n)} \varphi(x)$$
for $\varphi : \mathbb{S}^{d-1} \longrightarrow \mathbb{C}$ continuous.

<u>Prop.</u> The spherical polynomials span a <u>dense</u> subset of $\mathcal{C}(\mathbb{S}^{d-1})$; if $P \neq$ constant, then $\int_{\mathbb{S}^{d-1}} P \, d\mu_{unif} = 0$.

(cf. Iwaniec Th 9.1 + Stone–Weierstrass)

The "Weyl sum" above for $\varphi = P$ spherical homogeneous is $1 = \int \varphi \, d\mu_{Leb}$ if $P = 1$.

If P has degree $r \geq 1$ then
$$\frac{1}{\rho_d(n)} \sum_{x \in \mathcal{E}(n)} \varphi(x) = \frac{1}{\rho_d(n)} \sum_{Q(m) = n} P\left(\frac{m}{\sqrt{n}}\right)$$
$$= \frac{1}{n^{r/2}} \frac{1}{\rho_d(n)} \sum_{Q(m) = n} P(m)$$

~~so that~~ $\cancel{n^{1/2} \sum_{Q(m)} P(}$

$$= \frac{1}{\rho_r(n)} \frac{1}{n^{\nu/2}} \times \text{Fourier coefficient at } n$$

of <u>cusp form</u> $\theta(z; P)$

~~By the~~ The cusp form $\theta(z;P)$ has weight

$$k = \frac{1}{2} + \nu$$

so the Hecke bound gives

$$\frac{1}{\rho_r(n)} \sum_{x \in \mathcal{E}(n)} \varphi(x) = O\left(\frac{1}{\rho_r(n)} \frac{1}{n^{\nu/2}} n^{\frac{k}{2}} \right)$$

$$= O\left(\frac{1}{\rho_r(n)} n^{r/4} \right)$$

$$= O\left(n^{1 - r/4} \right)$$

For $r \geq 4$, this tends to 0. For $r = 4$, it <u>barely fails</u> ...

In the next chapter we will see a result of Rankin and Selberg that ~~solve~~ provides a better bound for Fourier coefficients of cusp forms. This will imply equidistribution also for $r = 4$...

<u>Th.</u> (Linnik). If $r \geq 4$, the points $\mathcal{E}(n)$, $n \to \infty$, become uniformly distributed on S^{r-1}.