6 - "Newforms" / Primitive forms

For k, q, χ general, there is no direct analogue of the basis $\mathcal{H}_k$ in $S_k(q,\chi)$. The problem is that it is not true that $T(n)$ is always normal. More precisely:

Lemma. (Hecke) For $(n,q)=1$, $T(n)$ is normal on $S_k(q,\chi)$, in fact

$$T(n)^* = \overline{\chi(n)}\, T(n)$$

So there is a basis of eigenfunctions of all $T(n)$ with $(n,q)=1$. But, this is not unique, because different f_i's may have the same eigenvalues.

Atkin-Lehner found the right way to work around this. They defined a subspace

$$S_k^{new}(q,\chi) \subset S_k(q,\chi)$$

which is preserved by all $T(n)$, and showed that multiplicity one holds in the "new" space. They also showed how to understand $S_k(q,\chi)$ from the new space and spaces $S_k(d,\chi')$ where $d \mid q$.

Elements of $S_k^{new}(q,\chi)$ which are eigenf. of $T(n)$ for $(n,q)=1$ are called newforms.

In particular, if $f \in S_k^{new}(q,\chi)$ is eigenfunction of all $T(n)$, $(n,q)=1$, it is <u>also</u> one of $T(n)$, $n \geq 1$: by commutativity, f and $T(m)f$ have the same $T(n)$-eigenvalues for $(n,q)=1$, hence are proportional.

Similarly, it follows that $\tilde{f} = f|_k \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$ is in $S_k^{new}(q,\bar{\chi})$ if f is a newform, and then that

$$\tilde{f} = \eta \sum_{n \geq 1} \overline{a_f(n)}\, e(nz)$$

for some η of modulus 1. So the functional equation relates $L(f,s)$ to $L(\bar{f},s)$ where

$$\bar{f}(z) = \sum \overline{a_f(n)}\, e(nz),$$

in particular

$$L(\bar{f},s) = \prod_p \frac{1}{1-\overline{\lambda_f(p)}\,p^{-s} + \chi(p)\,p^{k-1-2s}}$$