If $\operatorname{Im} gz < \operatorname{Im} z$, we replace (z, g) by (gz, g^{-1}) [using $g.z \in D$].

Now using this we conclude that in fact

$$\Gamma = SL_2(\mathbb{Z})$$

Indeed, let $g \in SL_2(\mathbb{Z})$. Then $g.(2i) \in \mathbb{H}$ so by (i) [in strong form] we find $h \in \Gamma$ s.t. $h.g.(2i) \in D$. Since $2i \in \mathring{D}$, the property (ii) implies $hg = \pm \mathrm{Id}$, so $g \in \Gamma$.

$\square$

3 _ First examples of modular forms

The proposition shows that it is easy to construct functions satisfying the modular transformation rules of weight k if we don't insist on any regularity property: just define $f(z)$ for $z \in D$, making sure that the "boundary cases" fit [e.g. $f(-\frac{1}{2}+iy) = f(\frac{1}{2}+iy)$] and then extend by "periodicity".

If one wants holomorphic or meromorphic functions, this is more complicated.

We will see various examples, and we start with a very general type of construction.

Basic principle: if a group G acts on a set X, if $f : X \longrightarrow \mathbb{C}$ is invariant under a subgroup $H < G$ [i.e. $f(h\cdot x) = f(h)$ for $h \in H$, $x \in X$] then, provided the sum makes sense, the function

$$P_f(x) = \sum_{g \in H\backslash G} f(gx)$$

is well-defined and G-invariant:

$$P_f(gx) = P_f(x), \qquad g \in G,\ x \in X$$

[Proof – first note that $f(g\cdot x)$ makes sense for $g \in H\backslash G$ since replacing g by hg gives

$$f(hg\cdot x) = f(h\cdot(gx)) = f(gx)$$

by assumption. Then for $\gamma \in G$

$$P_f(\gamma\cdot x) = \sum_{g \in H\backslash G} f(g\cdot(\gamma x))$$

$$= \sum_{g \in H\backslash G} f(g\gamma\cdot x)$$

and note that $g \longmapsto g\gamma$ permutes the classes in $H\backslash G$.]

We will now take $G = SL_2(\mathbb{Z})$

$$H = \left\{ \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$$

and $\quad f(z) = e(mz)$

where $\begin{cases} e(z) = e^{2i\pi z} \\ m \in \mathbb{Z} \end{cases}$ $\quad$ (note $|e(z)| = e^{-2\pi y}$)

Note $\quad f(z+n) = f\left(\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot z\right)$

$$\begin{aligned} &\quad \| \\ &e(mz + mn) \\ &\quad \| \\ &e(mz) \end{aligned} \qquad \text{since } e(k) = 1 \text{ for } k \in \mathbb{Z}$$

Since we are looking for weight k functions, we adjust the basic principle and define

$$P_{m,k}(z) = \sum_{g \in H \backslash SL_2(\mathbb{Z})} f(g \cdot z)(cz+d)^{-k}$$

<u>Claim:</u> if this series converges locally uniformly absolutely, then it is a modular form of weight k.

This is because of the same basic averaging argument, since

$$f(g \cdot z)(cz+d)^{-k} = (f|_k g)(z)$$

for the function f above.(*) The locally uniform convergence then implies that the function $P_{m,k}$ thus defined is in fact holomorphic on $\mathbb{H}$.

(*) One will get $(P_{m,k}|_k g)(z) = P_{m,k}(z)$, which is the modular transformation.

Proposition – If $m \geqslant 0$ and $k \geqslant 3$ then the series $P_{m,k}$ converges absolutely and locally uniformly on $\mathbb{H}$.

Note – Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$, any weight k form must satisfy

$$f(z) = (-1)^k f(z)$$

for all $z \in \mathbb{H}$, so k must be even if f is non-zero.

Proof. First note that

$$|e(mz)| = e^{-2\pi m \operatorname{Im}(z)} \leqslant 1$$

if $m \geqslant 0$ and $z \in \mathbb{H}$, so the statement will hold for all $m \geqslant 0$ if it does for $m = 0$, which we now assume. The following lemma gives some control of the cosets $H \backslash SL_2(\mathbb{Z})$.

Lemma – The map

$$\alpha \begin{cases} SL_2(\mathbb{Z}) \longrightarrow \mathbb{Z}^2 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c, d) \end{cases}$$

gives a well-defined map

$$\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\} \Big\backslash SL_2(\mathbb{Z}) \overset{\tilde{\alpha}}{\hookrightarrow} \mathbb{Z}^2$$

with image the set of coprime integers.

Proof. Note that

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$$

so $\tilde{\alpha}$ is well-defined. If $\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$

Then note that

$$\begin{aligned} ad - bc &= a'd - b'c \\ \Leftrightarrow \quad d(a-a') &= c(b-b') \end{aligned}$$

and since c, d are coprime (from $ad-bc=1$)

we get $\begin{cases} d \mid b-b', & \text{say} \quad b = b' + kd \\ c \mid a-a', & \text{say} \quad a = a' + lc \end{cases}$

Then $d(a-a') = dlc = ckd = c(b-b')$ so

$k = l$ and $\begin{cases} a = a' + kc \\ b = b' + kd \end{cases}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \bmod \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}\right\}$.

This shows that $\tilde{\alpha}$ is injective. Now $ad-bc=1$ shows that the image of $\tilde{\alpha}$ is included in the set of pairs of coprime integers. Conversely if c, d are coprime, then we know that there exist m, n in $\mathbb{Z}$ with

$$mc + nd = 1$$

so $\begin{pmatrix} n & -m \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $\alpha\begin{pmatrix} n & -m \\ c & d \end{pmatrix} = (c, d)$

$\square$

This lemma shows that

$$\sum_{g \in H \backslash SL_2(\mathbb{Z})} |cz+d|^{-k}$$

$$\leq \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{|cz+d|^k} .$$

<u>Lemma</u>. For $\sigma > 2$ and $z \in \mathbb{H}$, the series

$$\sum_{(c,d)\neq(0,0)} \frac{1}{|cz+d|^k}$$

converges absolutely and locally uniformly.

Assuming this, the Proposition follows immediately. ∎

<u>Proof</u>. We <u>claim</u> that for any $z \in \mathbb{H}$, we can find $\alpha(z) > 0$, $\beta(z) > 0$ such that α, β are locally bounded on $\mathbb{H}$ and

$$(*)\quad \begin{cases} |cz+d| \leq \alpha(z)\max(|c|,|d|) \\ \max(|c|,|d|) \leq \beta(z)\,|cz+d| \end{cases}$$

for all $(c,d) \in \mathbb{R}^2$.

Indeed, note that both $(c,d) \longmapsto \max(|c|,|d|)$ and $(c,d) \longmapsto |cz+d|$ are <u>norms</u> on $\mathbb{R}^2$ (the second because $z \in \mathbb{H}$ so $cz+d=0 \Rightarrow (c,d)=(0,0)$).

So the existence of $\alpha(z)$, $\beta(z)$ follows from the equivalence of norms on $\mathbb{R}^2$. To see that α, β are locally bounded, recall that for two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on $\mathbb{R}^d$ we have

$$\|v\|_2 \leq c\,\|v\|_1 \quad , \quad \forall\, v \in \mathbb{R}^d$$

for

$$c = \sup_{\|v\|_1 = 1} \|v\|_2 = \max_{\|v\|_1 = 1} \|v\|_2$$

So we can take

$$\begin{cases} \alpha(z) = \max\limits_{\max(|c|,|d|)=1} |cz+d| \\[2ex] \beta(z) = \max\limits_{|cz+d|=1} \max(|c|,|d|) \end{cases}$$

If z varies in a compact subset of $\mathcal{H}$, then these are bounded by basic properties of continuous functions on compact sets. Thus the claim is proved.

We can now finish the proof of the Lemma:

$$\sum_{(c,d)\neq(0,0)} \frac{1}{|cz+d|^\sigma} \leq A(z) + B(z)$$

where

$$A(z) = \frac{r_z(0)}{\min\limits_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} (|cz+d|)^\sigma}$$

$$B(z) = \sum_{n\geq 1} \frac{r_z(n)}{n^\sigma}$$

where

$$r_z(n) = \left| \{ (c,d) \in \mathbb{Z}^2 \mid n \leq |cz+d| < n+1 \} \right|$$

The Claim implies that

$$r_z(n) = O(n) \qquad \text{for } n \geq 1$$

uniformly in compact sets of $\mathbb{H}$ and

$$r_z(0) = O(1)$$

. Finally

$$\min_{(c,d) \in \mathbb{Z}^2 - \{(0,0)\}} \left(|cz+d| \right)$$

is also bounded from below uniformly locally because $\mathbb{Z} \oplus \mathbb{Z}z$ is discrete in $\mathbb{R}^2$.

So the series above is dominated by (locally uniformly)

$$c \sum_{n \geq 1} \frac{n}{n^\sigma}$$

for some $c > 0$, which converges for $\sigma > 2$.

$\square$