Chapter V

(cf. Iwaniec Chapter 13)

Introduction to the Langlands Program and elliptic curves

1 - Rankin - Selberg L-functions

(This last chapter will contain fewer detailed proofs...)

To improve on the bound $|a_f(n)| \ll n^{k/2}$ for $f \in S_k(q, \chi)$, where $a_f(n)$ are the Fourier coefficients at ∞, we will improve on the estimate

$$\sum_{n \leq X} |a_f(n)|^n \ll X^k$$

from which we deduced it.

Theorem - (Rankin, Selberg, ~ 1940)

Let $q \geq 1$ be an integer, χ Dirichlet character mod q, $f \in S_k(q, \chi)$, $(a_f(n))_{n \geq 1}$ its Fourier coefficients at ∞. For $X \geq 1$, we have

$$\sum_{n \leq X} |a_f(n)|^2 = c\, X^k + O(X^{k-1+3/5})$$

and therefore

$$|a_f(n)| \ll X^{\frac{k-1}{2} + \frac{3}{10}} \quad \text{for } n \geq 1.$$

We will explain the proof for $q = 1$, which avoids many complications. This will also determine the value of c. So from now on, we assume $q = 1$.

To prove this, we use the basic method for averages of arithmetic functions based on the generating Dirichlet series: we study

$$D(s) = \sum_{n\geq 1} \frac{|a_n|^2}{n^s}$$

and its analytic properties. Note that, at least, we have absolute, locally uniform, convergence as soon as $\mathrm{Re}(s) > k+1$. But we need analytic continuation beyond this region to do better.

Theorem. – (Rankin; Selberg)

(1) $D(s)$ admits a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=k$ with residue

$$\mathrm{Res}_{s=k} D(s) = \frac{3}{\pi} \|f\|^2 \frac{(4\pi)^k}{\Gamma(k)}$$

(2) More precisely, the function

$$L(s) = \zeta(2s-2k+2)\, D(s)$$

("Rankin-Selberg L-function")

admits meromorphic continuation to $\mathbb{C}$ with a unique simple pole at $s=k$ with residue

$$\mathrm{Res}_{s=k} L(s) = \frac{3}{\pi} \|f\|^2 \frac{(4\pi)^k}{\Gamma(k)} \zeta(2)$$

and with polynomial growth in vertical strips.

(3) [Selberg] If f is a primitive form then

$$L(s) = \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})^2(1-\beta_p^2 p^{-s})}$$

for $\mathrm{Re}(s) > k+1$, where α_p, β_p are such that

$$L(f,s) = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}$$

(4) We have

$$\Lambda(s) = \Lambda(2k-1-s)$$

where

$$\Lambda(s) = (2\pi)^{-2s}\,\Gamma(s-k+1)\,\Gamma(s) \times L(s)$$

In other words: we have here new L-function with the same properties as those of Dirichlet ——— of primitive characters, or primitive cusp forms.

The proof of this theorem is based on the Rankin-Selberg integral formula:

Theorem – For $Re(s)$ large enough, we have the formula

$$D(s+k-1) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \int_F y^k\, |f(z)|^2\, E(z,s)\, \frac{dx\,dy}{y^2}$$

where $F \subset \mathbb{H}$ is the usual fundamental domain, and $E(z,s)$ is the non-holomorphic Eisenstein series defined by

$$E(z,s) = \sum_{g \in \overline{B}\backslash SL_2(\mathbb{Z})} Im(gz)^s,$$

$\left(\overline{B} = \left\{ \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}\right)$

which converges absolutely for $Re(s) > 2$.

Proof - Recall that

$$\mathrm{Im}\, g z = \frac{\mathrm{Im}\, z}{|cz+d|^2}$$

so the terms have modulus

$$|(\mathrm{Im}\, g z)^s| = \frac{\mathrm{Im}(z)^\sigma}{|cz+d|^{2\sigma}} \quad , \sigma = \mathrm{Re}(s)$$

which defines an (absolutely / locally convergent / uniformly) series for $2\sigma > 2$, as we saw when proving the convergence of holomorphic Eisenstein series.

Now we compute the integral: for $\mathrm{Re}(s)$ large

$$\int_{SL_2(\mathbb{Z})\backslash H} y^k |f(z)|^2 E(z,s) \frac{dx\,dy}{y^2}$$

$$= \int_{SL_2(\mathbb{Z})\backslash H} y^k |f(z)|^2 \sum_{g \in \overline{H}\backslash SL_2(\mathbb{Z})} \mathrm{Im}(gz)^s \frac{dx\,dy}{y^2}$$

$$= \sum_{g \in \overline{H}\backslash SL_2(\mathbb{Z})} \int_F y^k |f(z)|^2 \mathrm{Im}(gz)^s \, d\mu(z)$$

where $d\mu = \frac{dx\,dy}{y^2}$ is $SL_2(\mathbb{R})$-invariant

$$= \sum_{g \in \overline{H}\backslash SL_2(\mathbb{Z})} \int_{g^{-1}F} y^k |f(z)|^2 \, y^s \, d\mu(z)$$

since $y^k |f(z)|^2$ is $SL_2(\mathbb{Z})$-invariant

$$= \int_{\substack{\text{fund. domain} \\ \text{for } \overline{H}\backslash H}} y^{k+s} \, |f(z)|^2 \, \frac{dx\,dy}{y^2}$$

$$= \int_0^1 \int_0^{+\infty} \sum_{n,m\geq 1} a_f(n)\overline{a_f(m)} \, e(nz)e(m\bar{z}) \, y^{k+s} \frac{dx\,dy}{y^2}$$

$$\cancel{= \int_0^{+\infty} y^{k+s-1} \Big(\sum_{n,m} \int_0^1 e(z(n-m)) \Big) \frac{dy}{y}}$$

$$= \int_0^{+\infty} y^{k+s-1} \Big(\int_0^1 \sum_{n,m} a_f(n)\overline{a_f(m)} \, e(nz)\,e(m\bar{z}) \, dx \Big) \frac{dy}{y}$$

$$= \int_0^{+\infty} y^{k+s-1} \sum_{n,m\geq 1} a_f(n)\overline{a_f(m)} \underbrace{\Big(\int_0^1 e(nz)e(m\bar{z}) \, dx \Big)} \frac{dy}{y}$$

$$= \int_0^1 e^{-2i\pi(n+m)y} e^{2i\pi(n-m)x} \, dx = \begin{cases} 0, & n \neq m \\ e^{-4\pi n y}, & n = m \end{cases}$$

$\rightarrow$

$$= \sum_{n\geq 1} |a_f(n)|^2 \int_0^{+\infty} y^{k+s-1} e^{-4\pi n y} \frac{dy}{y}$$

which gives the result by change of variable from the definition of the Γ-function.

It is here straightforward to justify the exchange of sums and integrals for $\mathrm{Re}(s)$ large enough, in view of the absolute convergence of all relevant series.

$\square$

The Rankin-Selberg integral formula transfers the problem of analytic continuation to that of the Eisenstein series $E(z,s)$, as <u>functions of s</u>. [<u>Note</u>: these are called <u>non-holomorphic</u> because they are not holomorphic <u>as functions of $z \in \mathbb{H}$</u> ...]

<u>Theorem</u> - The Eisenstein series admits meromorphic continuation to a meromorphic function on $\mathbb{C}$, for each $z \in \mathbb{H}$ [or as "vector-valued" holomorphic functions].

This is deduced from a more precise result, with once more a functional equation:

<u>Proposition</u> - The Eisenstein series $E(z,s)$ admits the Fourier expansion

$$\Theta(s)\, E(z,s) = \Theta(s)\, y^{s} + \Theta(1-s)\, y^{1-s} + 4\sqrt{y} \sum_{n \neq 0} \sigma_{s-\frac{1}{2}}(|n|)\, K_{s-\frac{1}{2}}(2\pi |n| y)\, e(nx)$$

where

$$\Theta(s) = \pi^{-s}\, \Gamma(s)\, \zeta(2s),$$

$$\sigma_w(n) = \sum_{ad=n} \left(\frac{a}{d}\right)^w \quad , \quad w \in \mathbb{C}$$

and

$$K_w(y) = \frac{1}{2}\int_0^{+\infty} e^{-\frac{y}{2}(t+t^{-1})}\, t^{w-1}\, dt$$

for $y>0$ ("K - Bessel function, of the second kind)

<u>Note</u>. The Fourier expansion can be expressed also in terms of the Whittaker function

$$W_w(z) = 2\sqrt{|y|}\, K_w(2\pi|y|)\, e(nx)$$

which oscillates and in fact can be shown to be $\sim e(z) = e^{-2\pi y} e(x)$ as $z \to \infty$ in $\mathbb{H}$.

<u>Corollary</u>. The Eisenstein series ~~satisfies the~~ ~~functi~~ is meromorphic on $\mathbb{C}$ and satisfies the functional equation

$$\theta(s)\; E(z,s) = \theta(1-s)\; E(z,1-s)$$

<u>Proof</u>. That $E(z,s)$ is meromorphic follows from the fact that $\theta(s)$ is, as a consequence of the analytic continuation of the Riemann zeta function, and from the fast decay of $K_{s-\frac{1}{2}}(2\pi|n|y)$ as $|n| \to +\infty$.

The functional equation is then apparent from the symmetry

$$\sigma_w(n) = \sigma_{-w}(n).$$

for $n \geq 1$, $w \in \mathbb{C}$. $\square$

From the Rankin - Selberg formula

$$D(s+k-1) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \int_F y^k |f(z)|^2 E(z,s) \frac{dx\,dy}{y^2}$$

and the fact that $f \to 0$ exponentially fast as $z \to \infty$ on F, we deduce that $D(s)$ is meromorphic on $\mathbb{C}$, and then that the function

$$(4\pi)^{-s-k+1} \Gamma(s+k-1)\, \pi^{-s} \Gamma(s)\, \zeta(2s)\, D(s+k-1)$$

is invariant by the operation $s \longleftrightarrow 1-s$.

This means after some computation that

$$\Lambda(s) = (2\pi)^{-2s} \Gamma(s)\Gamma(s-k+1)\,\zeta(2(s-k+1))\,D(s)$$

is invariant by $s \longleftrightarrow 2k-1-s$.

We now identify the poles and residues for $\Lambda(s)$ ($D(s)$ will have more poles, related to zeros of the zeta function); again these are provided by the poles of $E(z,s)$, and by symmetry we need only consider $Re(s) \geq \frac{1}{2}$. Then $\Theta(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ does not vanish, so the only poles can arise from $\frac{\Theta(1-s)}{\Theta(s)} y^{1-s}$ in the Fourier expansion, in other words from poles of

$$\Theta(1-s) = \pi^{-(1-s)} \Gamma(1-s)\, \zeta(2-2s)$$

This is the completed ζ function at $2-2s$ which has poles at 0 and 1, corresponding to $s = 1$ or $s = \frac{1}{2}$ in this case. At $s = 1/2$, the pole is cancelled out by the zero of $\frac{1}{\theta(s)}$ [coming from the pole of $\zeta(s)$ at $s=1$] but the pole at $s=1$ remains. The residue is

$$\operatorname{res}_{s=1} E(z,s) = y^{1-1} \cdot \operatorname{res}_{s=1} \left(\frac{\theta(1-s)}{\theta(s)} \right)$$

$$= \frac{1}{1/\pi \, \Gamma(1) \zeta(2)} \underbrace{\operatorname{res}_{s=1} \pi^{-(1-s)} \Gamma(1-s) \zeta(2-2s)}_{=2}$$

by functional equation and $\operatorname{res}_1 \zeta = 1$

$\Rightarrow$

$$\operatorname{res}_{s=1} E(z,s) = \frac{1}{\frac{1}{\pi} \cdot 1 \cdot \frac{\pi^2}{6} \cdot 2} = \frac{\pi}{3}.$$

We <u>conclude</u> that $\Lambda(s)$ has a simple pole at $s = k$ with residue

$$\frac{3}{\pi} \underbrace{\int_F y^k |f|^2 \frac{dx\,dy}{y^2}}_{\|f\|^2}$$

<u>Note:</u> one can check that

$$\frac{\pi}{3} = \int_F \frac{dx\,dy}{y^2}.$$

Suppose finally that $f \in \mathcal{H}_k$ is a <u>primitive</u> form. Then $a_f(n)$ is real (because $T(n)$ is self-adjoint) and $n \mapsto a_f(n)^2$ is multiplicative, so

$$\sum_{n\geq 1} a_f(n)^2 n^{-s} = \prod_p \sum_{\nu \geq 0} a_f(p^\nu)^2 p^{-\nu s}$$

<u>Lemma</u> (Selberg) – Let α, β be complex numbers and $(\alpha_n)_{n\geq 0}$ defined by

$$\frac{1}{(1-\alpha X)(1-\beta X)} = \sum_{n\geq 0} \alpha_n X^n$$

Then

$$\sum \alpha_n^2 X^n = \frac{1-(\alpha\beta)^2 X^2}{(1-\alpha^2 X)(1-\alpha\beta X)^2(1-\beta^2 X)}$$

Applied for each prime p with α_p, β_p such that

$$\sum a_f(p^\nu) X^\nu = \frac{1}{(1-\alpha_p X)(1-\beta_p X)}$$

(i.e. $\alpha+\beta = a_f(p)$, $\alpha\beta = p^{k-1}$, by the theory of Hecke operators), we deduce

$$\sum a_f(n)^2 n^{-s} = \prod_p \frac{1-p^{2((k-1)-s)}}{(1-\alpha_p p p^{-s})(1-p^{k-1-s})^2(1-\beta_p p^{-s})}$$

$\Rightarrow$

$$\underbrace{\zeta(2(s-k+1))\,D(s)}_{L(s)} = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-p^{k-1-s})^2(1-\beta_p p^{-s})}$$

2 - The briefest introduction to ideas of Langlands

By now, we have three examples of Dirichlet series sharing some properties:

(i) analytic continuation with very few poles

(ii) functional equation after multiplying by some "Γ factor"

(iii) Euler product with Euler factors of the type $\dfrac{1}{\prod_{i=1}^{d_p} (1-\alpha_i p^{-s})}$, where d_p is the same for all but finitely many primes

	p - factor	Γ - factor	Condition
$L(s,\chi)$	$\dfrac{1}{1-\chi(p)p^{-s}}$	$\pi^{-s/2}\Gamma(s/2)$ or $\pi^{-\frac{(s+1)}{2}}\Gamma(\frac{s+1}{2})$	χ primitive
$L(f,s)$	$\dfrac{1}{1-\lambda_f(p)p^{-s}+p^{k-1-2s}}$	$(2\pi)^{-s}\,\Gamma(s)$	$f \in H_k$
$L(f\times f,s)$	$\dfrac{1}{(1-\alpha_p^2 p^{-s})(1-p^{k-1-s})^2(1-\beta_p^2 p^{-s})}$	$(2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s+k-1)$	$f \in H_k$

Is there a pattern? Are there other examples?

~~In some sense~~ For instance, let $f \in H_k$, $f \neq 0$ and define α_p, β_p as usual. If we take same expression like

$$\prod_p (1-\alpha_p^a p^{-s})(1-\beta_p^b p^{-s}) \cdots (1-\alpha_p^{a'}\beta_p^{b'} p^{-s}) \cdots$$

does it have these good properties?

Or: what about

$$\sum a_f(n)^k n^{-s}, \quad f \in \mathcal{H}_k$$

for $k \geq 2$ arbitrary integers?

~~Langlands understood what is going on...~~

Or: since

$$L(f\times f, s) = \prod \frac{1}{(1-\alpha_p^2 p^{-s})(1-p^{k-1-s})^2(1-\beta_p^2 p^{-s})}$$

can we write this as

$$\zeta(s-k+1)^2 \, (\text{other good } L\text{-function})$$

or not?

Langlands understood what was going on in the 1960's.

The key point for the Rankin-Selberg L-function is that the denominator of the p-factor is

$$(1-\alpha_p^2 X)(1-p^{k-1}X)^2(1-\beta_p^2 X)$$

$$= \det(1 - X A_p \otimes A_p)$$

where

$$A_p = \begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix}$$

and we have here an "algebraic representation" ($g \mapsto g \otimes g$) of the group of 2×2 matrices: a group homomorphism

$$\begin{cases} GL_2(\mathbb{C}) \xrightarrow{\rho} GL_4(\mathbb{C}) \\ g \longmapsto g \otimes g \end{cases}$$

where the eigenvalues of $\rho(g)$ are

$$\alpha^2, \alpha\beta, \alpha\beta, \beta^2$$

if α, β are eigenvalues of g.

This is the pattern: if

$$\tau : GL_2(\mathbb{C}) \longrightarrow GL_N(\mathbb{C})$$

is such an homomorphism, then Langlands conjectured that $\prod_p \det(1 - p^{-s}\tau(A_p))^{-1}$

$\boxed{\text{should}}$ have all the good properties above.

Ex. $\tau : GL_2(\mathbb{C}) \longrightarrow GL_3(\mathbb{C})$

$$g \longmapsto \mathrm{Sym}^2(g)$$

$\Vert$

(action of g on $aX^2 + bXY + cY^2$ by linear change of variable)

$$\rightsquigarrow \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-p^{k-1}p^{-s})(1-\beta_p^2 p^{-s})} = \frac{L(f\times f, s)}{\zeta(s+k-1)}$$

should have these $\rightsquigarrow$ "symmetric square L-function"

So the Rankin - Selberg L-function can be divided by one zeta factor, but not two.

For the symmetric square, the analytic properties are due to Shimura. For $f \in H_k$ the whole set was proved quite recently only by Newton and Thorne, but generalization to other modular forms (f primitive of level q)

Here is a concrete corollary:

Cor. (Newton - Thorne; conj. by Sato-Tate, strategy by Serre)

Let $m \geq 0$ be an integer, $f \in H_k$.

We have

$$\frac{1}{\pi(x)} \sum_{p \leq x} \left(\frac{a_f(p)}{p^{\frac{k-1}{2}}} \right)^{2m} \xrightarrow[x \to \infty]{} c_m$$

where

$$c_m = \frac{1}{m+1} \binom{2m}{m}$$ is a Catalan number

$(1, 1, 2, 5, 14, \dots)$

This means that $\left\{ \frac{a_f(p)}{p^{(k-1)/2}} \;\middle|\; p \leq x \right\}$ becomes equidistributed for the Sato-Tate measure.

Here c_m appears as order of the pole at $s=1$ of the series

$$\sum_{n \geq 1} \left(\frac{a_f(n)}{n^{(k-1)/2}} \right)^{2m} n^{-s}.$$

In fact, Langlands went further: he explained why/how these Dirichlet series should have these properties: they should be L-function of analogues of modular forms for GL_m instead of GL_2.

In this simplest case, this means: for $f \in \mathcal{H}_k$

- there is a function

$$f_m : SL_n(\mathbb{R}) \longrightarrow \mathbb{C}$$

invariant under $SL_n(\mathbb{Z})$

- there is a family of Hecke operators for such functions, $T_n^{(m)}$

- f_m is an eigenfunction of all $T_n^{(m)}$ with eigenvalues equal to $Tr\, \tau(A_p)$

. and this f_m has an L-function, similar to Hecke's, which one shows has the expected analytic properties.

[Gelbart – Jacquet, 80's, for sym^2]

This is what Newton – Thorne proved; the last step (analytic properties of L-functions on SL_m) was already known for a long time (Godement, Jacquet, Langlands).

For $f \times f$, this is also true, but was only proved (with a form on SL_4) ~ 1999 by Ramakrishna.